Formal proofs and certified computation in Coq for solving the Table Maker's Dilemma

Érik Martin-Dorel

Postdoc in the Toccata team, Inria Saclay – Île-de-France, LRI

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Acknowledgements

The TaMaDi project of the ANR

Project leader: Jean-Michel Muller

Project members: Jean-Claude Bajard, Nicolas Brisebarre, Florent de Dinechin, Pierre Fortin, Mourad Gouicem, Stef Graillat, Guillaume Hanrot, Thibault Hilaire, Mioara Joldeș, Christoph Lauter, Vincent Lefèvre, Érik Martin-Dorel, Micaela Mayero, Marc Mezzarobba, Jean-Michel Muller, Andy Novocin, Ioana Pașca, Laurence Rideau, Damien Stehlé, Laurent Théry, Serge Torres.

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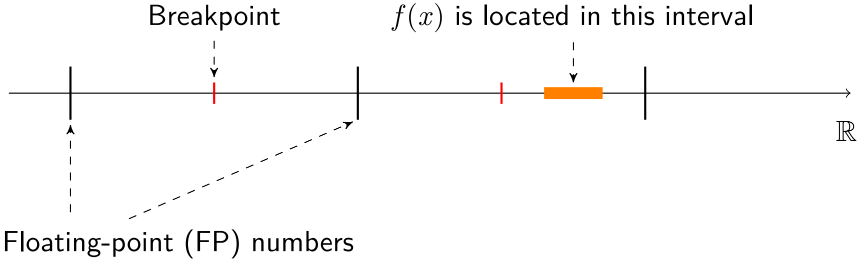
IEEE 754–2008 only recommends correct rounding for elementary functions $(\exp, \sin, \dots)$ $\Rightarrow$ solve the Table Maker's Dilemma for each function.

The Table Maker's Dilemma (TMD) in rounding-to-nearest



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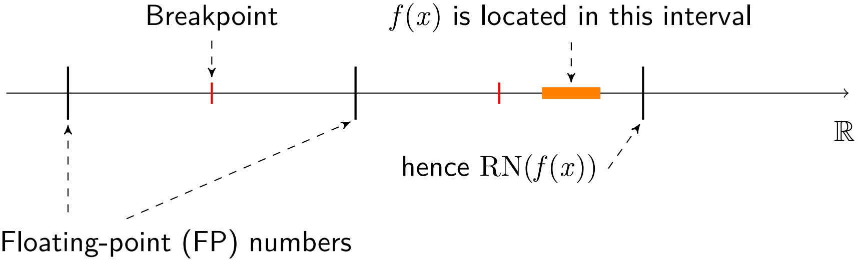

Breakpoint

$f(x)$ is located in this interval

$\mathbb{R}$

hence $\mathrm{RN}(f(x))$

Floating-point (FP) numbers

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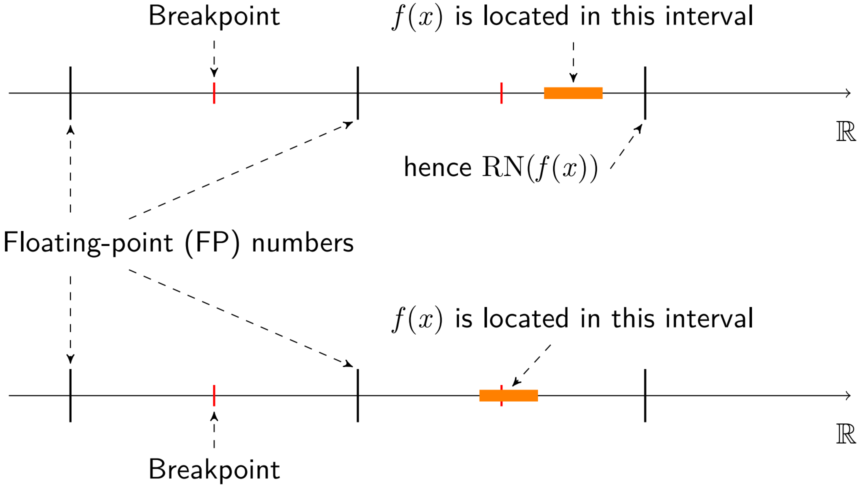

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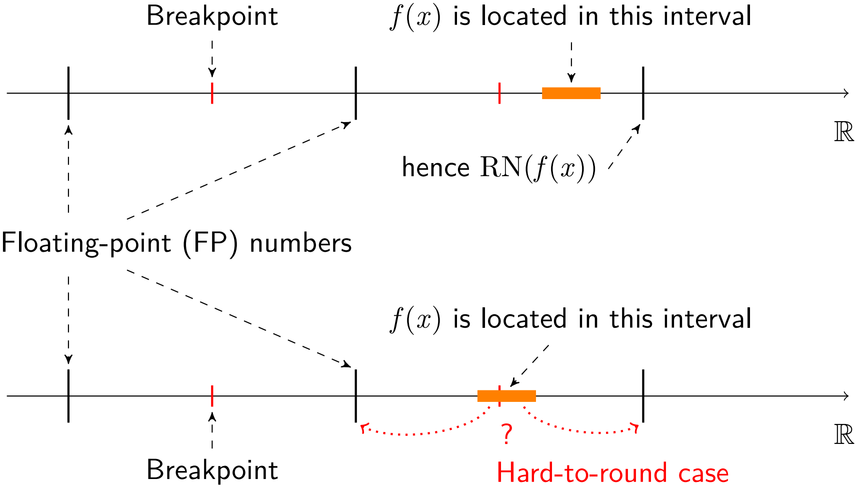

The Table Maker's Dilemma (TMD) (continued)

Solving the TMD $=$ find the hardest-to-round cases of f: the FP values x such that $f(x)$ is closest to a breakpoint without being a breakpoint.

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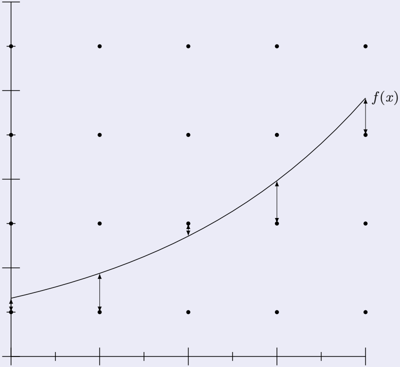
$$x = 9.407822313572878 \times 10^{-2}$$

$$\exp(x) = 1.098645682066338\ 5\ {\color{red}0000000000000000}\ 278\ldots$$

Computation of lists of hard-to-round cases

Finding all the hard-to-round cases of f over $\boldsymbol{I}$ with respect to $\epsilon > 0$

$$\left\{x \in \mathbb{F} \cap \boldsymbol{I} \;\middle/\; \left|\left(\frac{f(x)}{\mathrm{ulp}(f(x))} - \frac{1}{2}\right) \operatorname{cmod} 1\right| \leqslant \epsilon\right\}.$$



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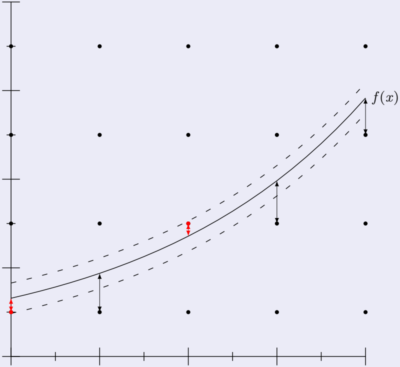

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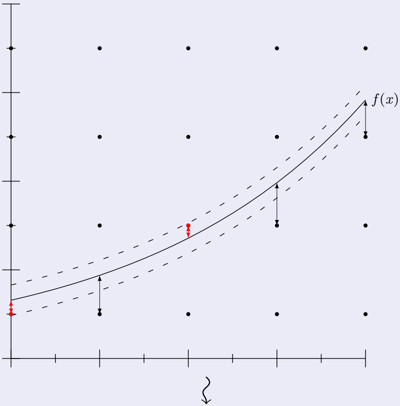

Integer Small Value Pbm $(P \in \mathbb{Z}[X], A, B, M \in \mathbb{Z}) = \{x \in [\![-A, A]\!] \,/\, |P(x) \operatorname{cmod} M| \leqslant B\}$.

The Stehlé–Lefèvre–Zimmermann (SLZ) algorithm

SLZ = Polynomial Approximation + Coppersmith's technique

+ Bivariate Hensel lifting

The Stehlé–Lefèvre–Zimmermann (SLZ) algorithm

Critical part of the algorithm

SLZ = Polynomial Approximation + Coppersmith's technique
+ Bivariate Hensel lifting

- Sophisticated algorithms with highly optimized implementations
- Rely on many tools and libraries (SAGE, MPIR, FLINT, fpLLL, . . .)
- Very long calculations (several years of CPU time)

The SLZ algorithm (continued)

TMD

First step: Turn the TMD into a problem involving integers

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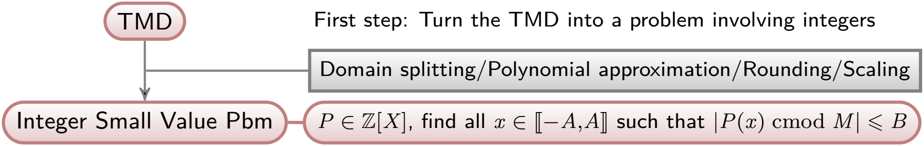

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First step: Turn the TMD into a problem involving integers

Domain splitting/Polynomial approximation/Rounding/Scaling

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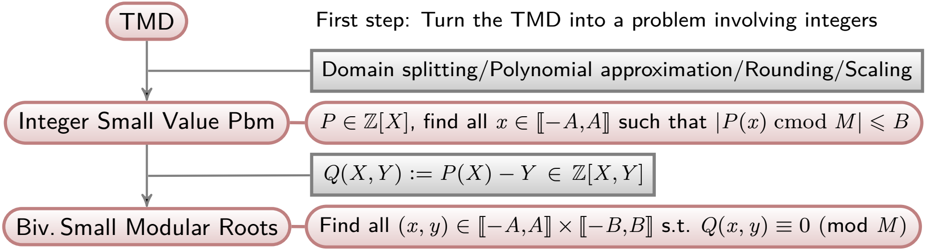

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$Q(X,Y) := P(X) - Y \ \in \mathbb{Z}[X,Y]$

Biv. Small Modular Roots

Find all $(x,y) \in \llbracket -A, A \rrbracket \times \llbracket -B, B \rrbracket$ s.t. $Q(x,y) \equiv 0 \pmod{M}$

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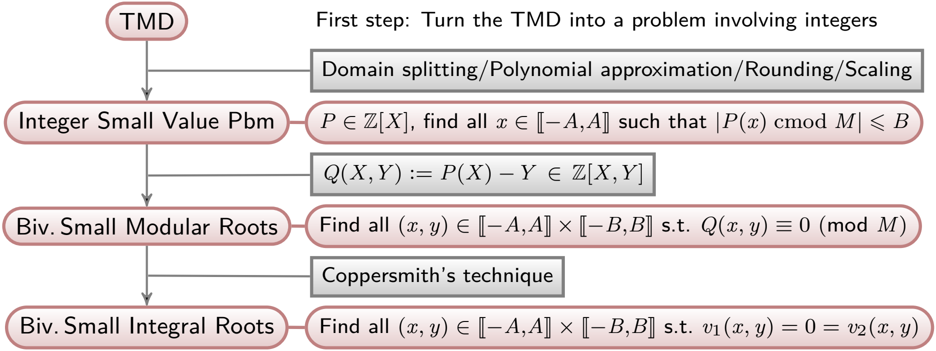

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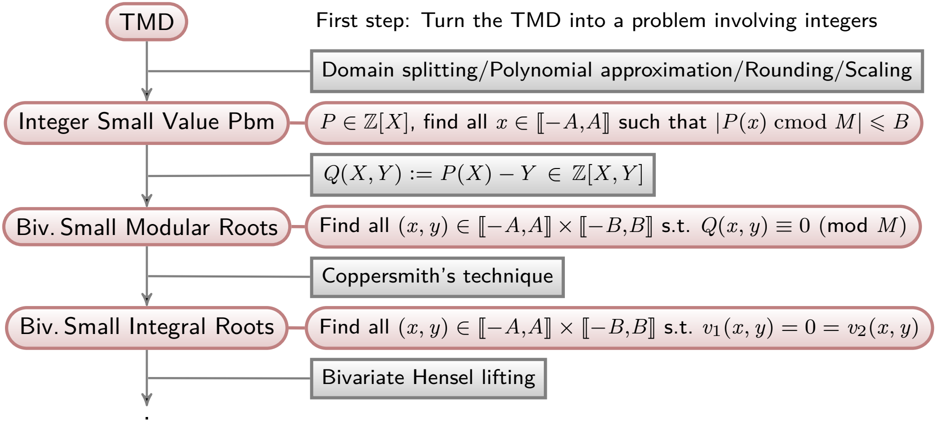

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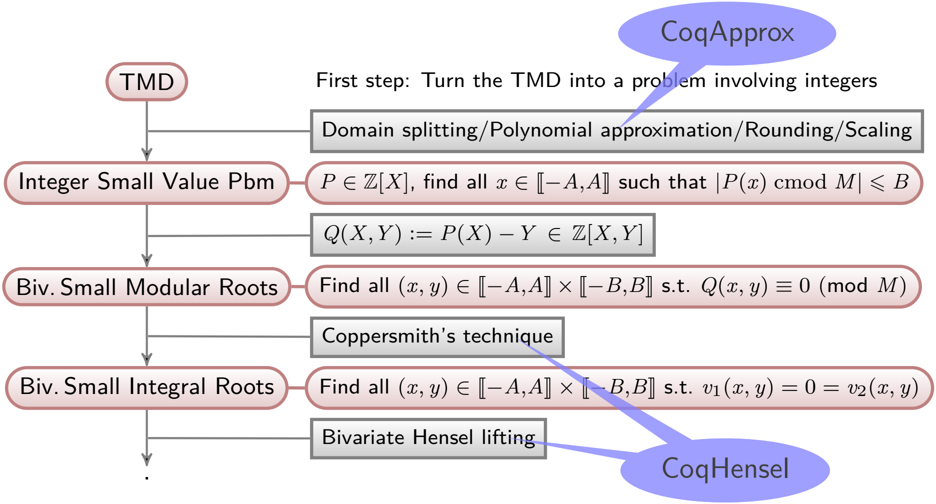

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Outline

CoqApprox: Context and motivations

Goal

Compute polynomial approximations of univariate functions along with certified error bounds: for a given function f over an interval[a] $\boldsymbol{I}$, compute P, ϵ and formally prove that $\forall x \in \boldsymbol{I},\ |f(x) - P(x)| \leqslant \epsilon$

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[a] Intervals are printed in bold.

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Example

For $f(x) = \sin x$ over $\boldsymbol{I} = [-1, 1]$ and a target accuracy of 2^{-400}, we can compute an order-80 Taylor expansion of f around 0 for the polynomial P and take $\epsilon = 1.79 \times 2^{-402}$.

CoqApprox: Mathematical setup

Data structure

A rigorous polynomial approximation (RPA) is a pair $(P, \mathbf{\Delta})$ where P is a polynomial in a given basis, and $\mathbf{\Delta}$ an interval. Typical examples of RPAs are Taylor Models (TMs) and Chebyshev Models (CMs).

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Methodology

1. For basic[a] functions: rely on the Taylor–Lagrange formula.
2. For composite functions, we define some "arithmetic rules" for addition, multiplication, composition, and division.
   E.g.: if $(P_1, \mathbf{\Delta}_1)$ is a TM of f_1 and $(P_2, \mathbf{\Delta}_2)$ is a TM of f_2, then $(P_1, \mathbf{\Delta}_1) \oplus (P_2, \mathbf{\Delta}_2) := (P_1 + P_2, \mathbf{\Delta}_1 + \mathbf{\Delta}_2)$ is a TM for $f_1 + f_2$.

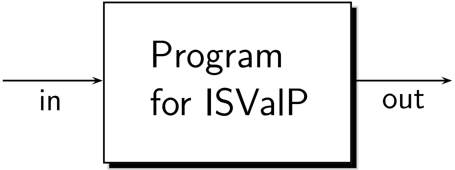
---

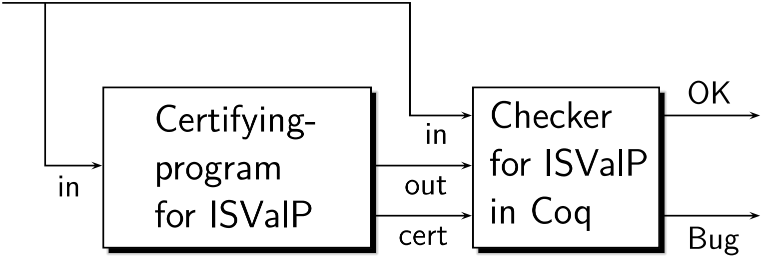
[a] We focus on D-finite (*aka* holonomic) functions, i.e., solutions of homogeneous linear ordinary differential equations with polynomial coefficients.

CoqApprox: Formalization and machine-checked proofs

- Libraries used: Ssreflect [MathComponents], CoqInterval [Melquiond]
- Efficiency: on the whole, the timings of the Coq implementation have the same order of magnitude as that of the C implementation provided in Sollya [Chevillard, Joldeș, Lauter]
- Sharp bounds: thanks to the implemented algorithm called Zumkeller's technique, the approximation of basic functions leads to sharp bounds in practice.

From SLZ to certificates for Integer Small Value Problems





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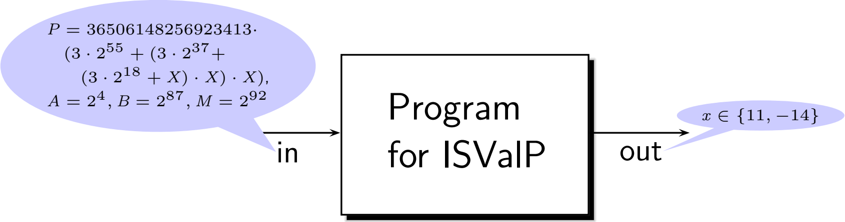


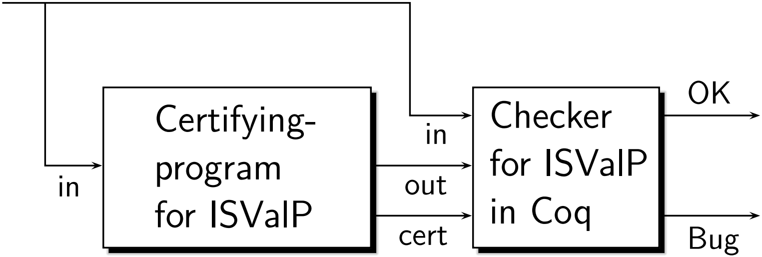
$P = 36506148256923413 \cdot (3 \cdot 2^{55} + (3 \cdot 2^{37} + (3 \cdot 2^{18} + X) \cdot X) \cdot X)$, $A = 2^4$, $B = 2^{87}$, $M = 2^{92}$

in

Program for ISValP

out

$x \in \{11, -14\}$



in

Certifying-program for ISValP

in

out

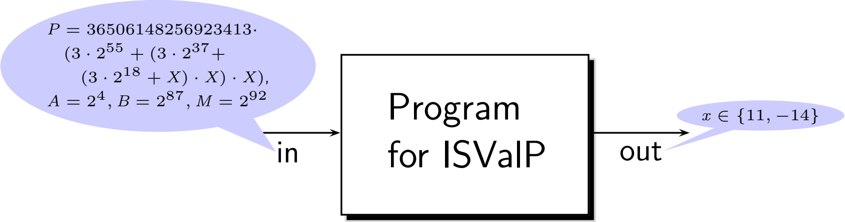
cert

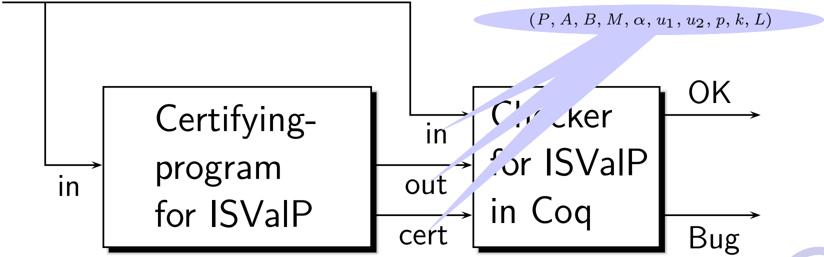
Checker for ISValP in Coq

OK

Bug

From SLZ to certificates for Integer Small Value Problems





A verified checker for the Integer Small Value Problem

Theorem

For any certificate $(P, A, B, M, \alpha, u_1, u_2, p, k, L)$ that is accepted, we have

$$\forall (x, y) \in [\![-A, A]\!] \times [\![-B, B]\!],\ P(x) \equiv y \pmod{M} \implies (x, y) \in L.$$

The Coq proof required the formalization of several mathematical notions:

- Taylor's theorem for bivariate polynomials,
- Hensel's lemma for pairs of bivariate polynomials,
- properties of the weighted norm-1 of a bivariate polynomial,

...

Challenges and methodology

Formalizing efficient computation in a proof assistant is often challenging.

- Approach by refinement: first, prove an abstract version of the algorithms, then refine them to effective implementations that are proved correct w.r.t. the abstract version.

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- Approach by certificates: rather than formally verifying the correctness of an optimized implementation of a complex algorithm such as LLL, generate some “logs” of its execution and check them independently.

Milestone

We can compute formally verified Taylor Models for the following D-finite functions: $x \mapsto \frac{1}{x}$, $\sqrt{\cdot}$, $\frac{1}{\sqrt{\cdot}}$, $\exp$, $\sin$, $\cos$,
and the algorithms to compute Taylor Models for composite functions (involving the operations $+$, $\times$, $\circ$, $\div$) have also been formally verified.

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Test-suite for CoqHensel:

- 4096 ISValP certificates to address an exponent of $\exp$ in binary64: $n := 53$, $n' := 300$, $\alpha := 13$, $\approx 100\,$MB of data. The generation of each certificate takes $\approx 140\,$s, and the verification in Coq takes $\approx 35\,$s (using `native_compute` with "`bigZ` $\times$ `bigN`" integers).

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- one ISValP certificate for $\exp$ in binary128: $n := 113$, $n' := 3000$, $\alpha := 6$. Verified by Coq in $1\,041$ s (vs. 56 s in Maple 15).

Future work

Short-term perspectives:

- Implement faster algorithms for the operations of polynomials.
- Combine CoqHensel & CoqApprox to devise a complete certificate checker for the TMD.
- Implement and prove more functions in CoqApprox $(\cosh, \tan, \dots)$

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- On-going works: formal proof of Lefèvre's algorithm.

End of the talk

Thank you for your attention!

The TaMaDi homepage:
https://tamadiwiki.ens-lyon.fr/

Certificates to address the Integer Small Value Problem

```
Record cert_ISValP :=
{ $P$ : {poly int}   (* hence $Q(X,Y) = P(Y) - X$ *)
; $A$ : nat          (* bound related to the TMD accuracy *)
; $B$ : nat          (* bound related to the domain range *)
; $M$ : nat          (* the modulo *)
; $\alpha$ : nat          (* the Coppersmith parameter *)
; $u_1$ : {bipoly int} (* in basis $M^{\alpha-i} \times Q^i(X,Y) \times Y^j$ *)
; $u_2$ : {bipoly int} (* in basis $M^{\alpha-i} \times Q^i(X,Y) \times Y^j$ *)
; $p$ : nat          (* prime used by Hensel lifting *)
; $k$ : nat          (* number of iterations *)
; $L$ : list (int * int * bool) (* list of solutions *)
}.

Definition check_ISValP : cert_ISValP -> bool.
```

Back