Searching for sinks of Hénon map using a multiple-precision GPU arithmetic library

Mioara Joldeș, Valentina Popescu, Warwick Tucker

RAIM 2013

Hénon attractor

Hénon Map

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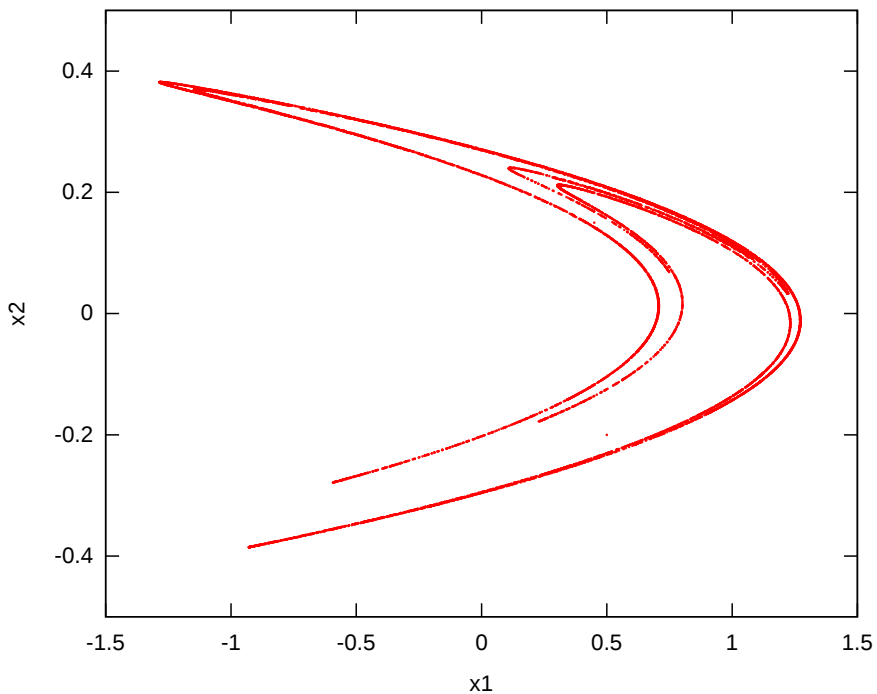

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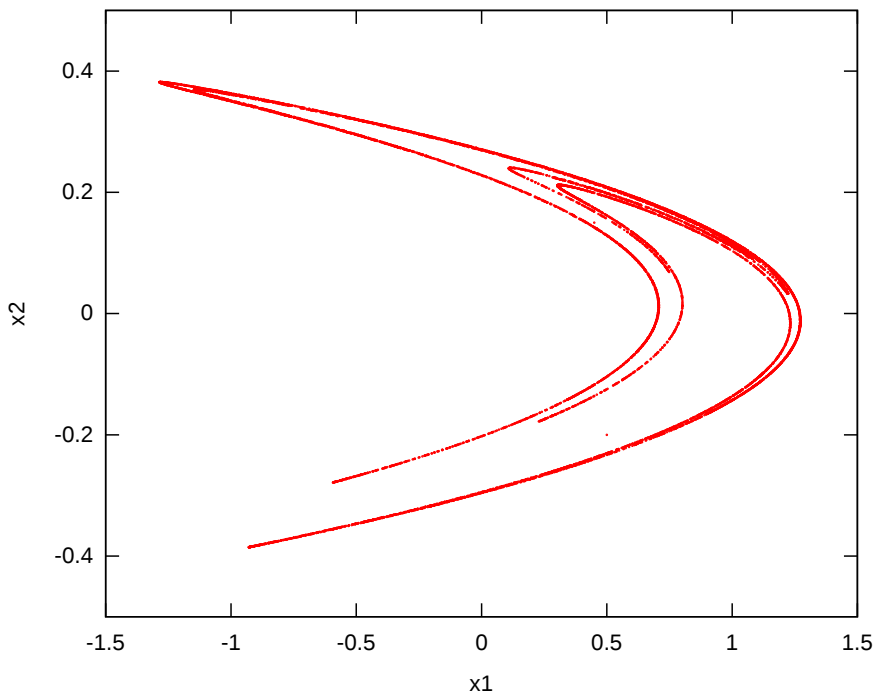

Open question: Is this a Strange Attractor ?

Some basic terminology

Orbit of x: $\Gamma(x) = \Gamma^+(x) \cup \Gamma^-(x)$

- Forward orbit:

$$\Gamma^+(x) := \{h^n_{a,b}(x), n \in \mathbb{N}\}$$

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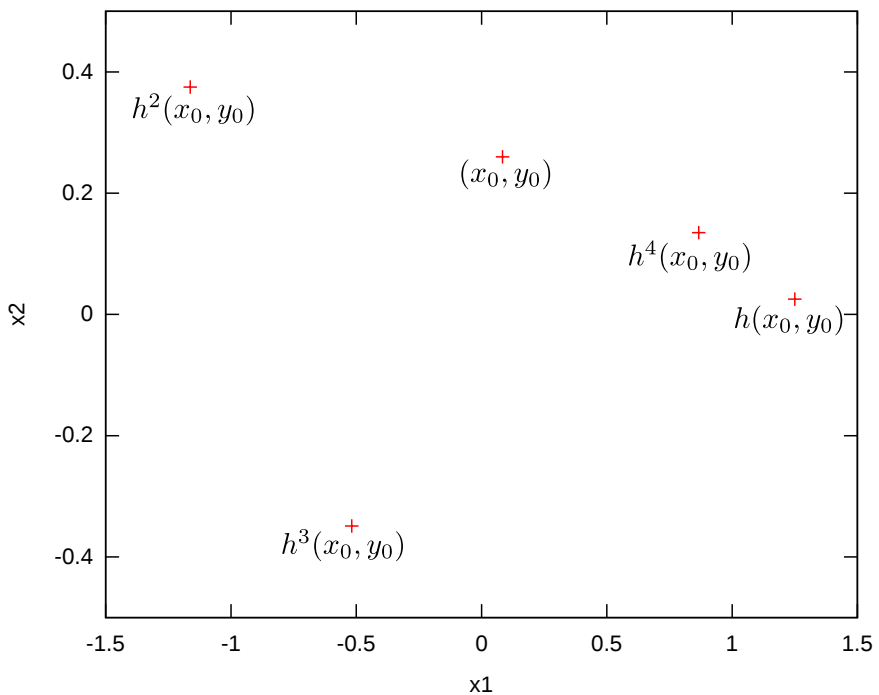

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Stable orbit $\Gamma(x)$

Let $d(x, y) = ||x - y||$ and $d(y, \Gamma(x)) = \inf_{z \in \Gamma(x)} d(y, z)$.

$\Gamma(x)$ is stable if given $\varepsilon > 0, \exists \delta > 0$ s.t. $d(h_{a,b}^n(y), \Gamma(x)) < \varepsilon$, $\forall n \in \mathbb{N}^*$ and $\forall y$ s.t. $d(y, \Gamma(x)) < \delta$.

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Asymptotically Stable orbit $\Gamma(x)$ (sink)

$\Gamma(x)$ is asymptotically stable if it is stable and (by choosing δ smaller if necessary) $d(h^n_{a,b}(y), \Gamma(x)) \to 0$ as $n \to \infty$.

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Basin of attraction

Union of orbits which converge towards the attractor.

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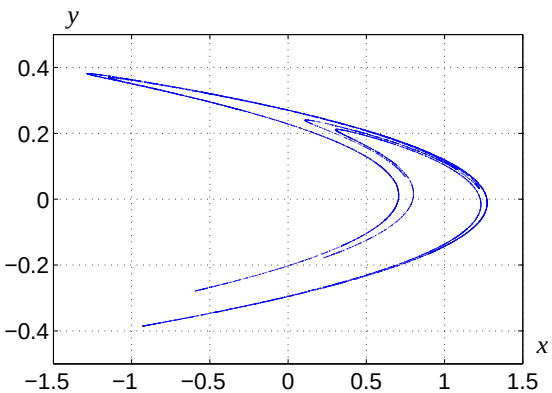

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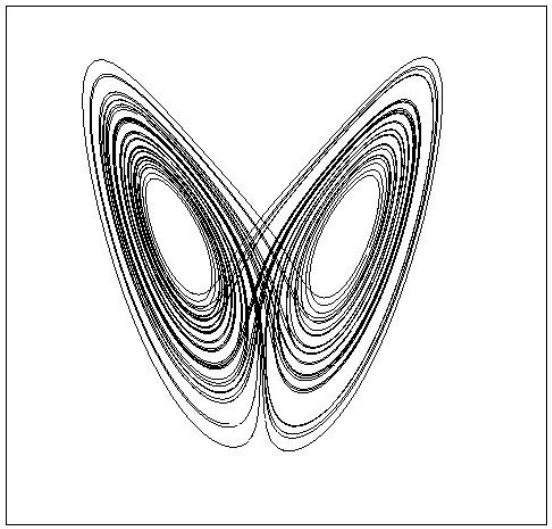

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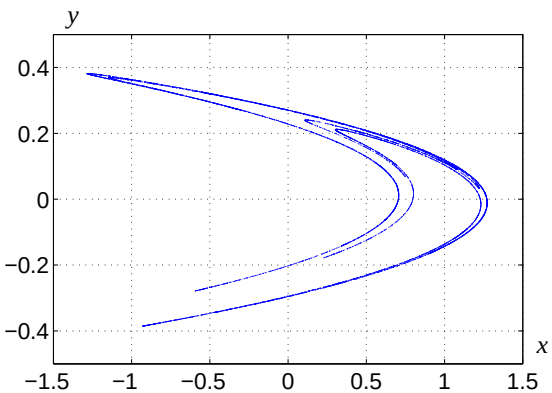

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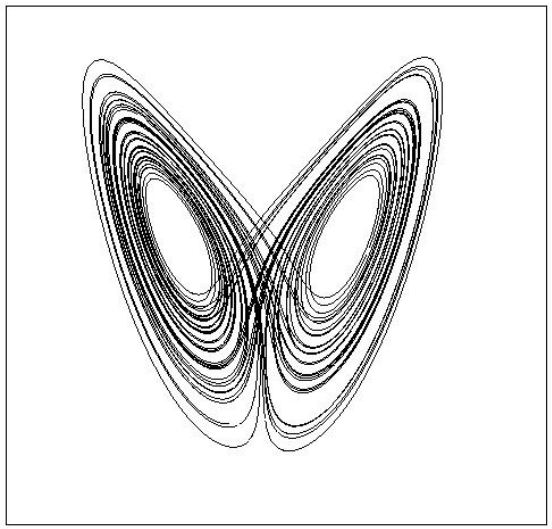

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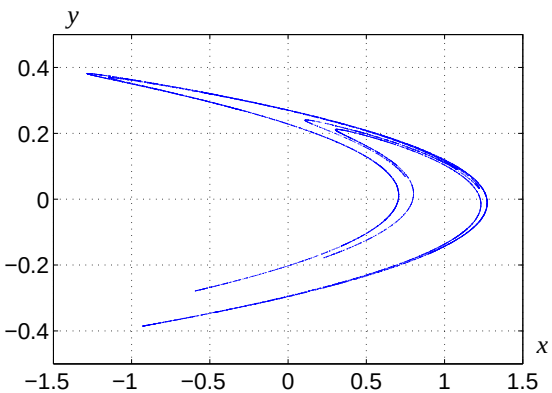

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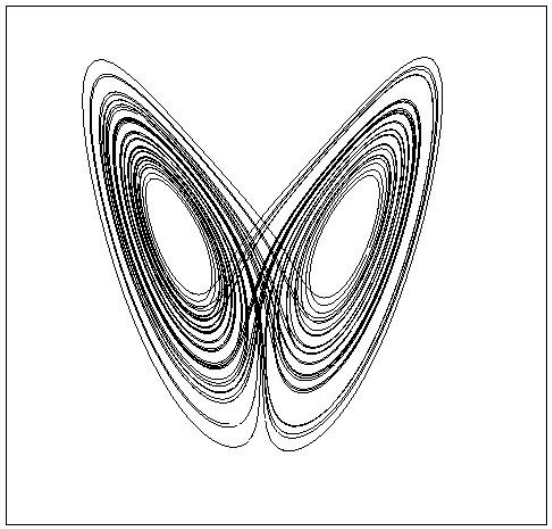

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- name coined by Takens and Ruelle $\simeq 1971$
- Properties:[Ruelle 2006]
  - chaotic dynamics e.g.
  - fractal dimension fig.
  - invariant in the sense that the system comes arbitrarily close to any point on the attractor independently of the initial conditions.

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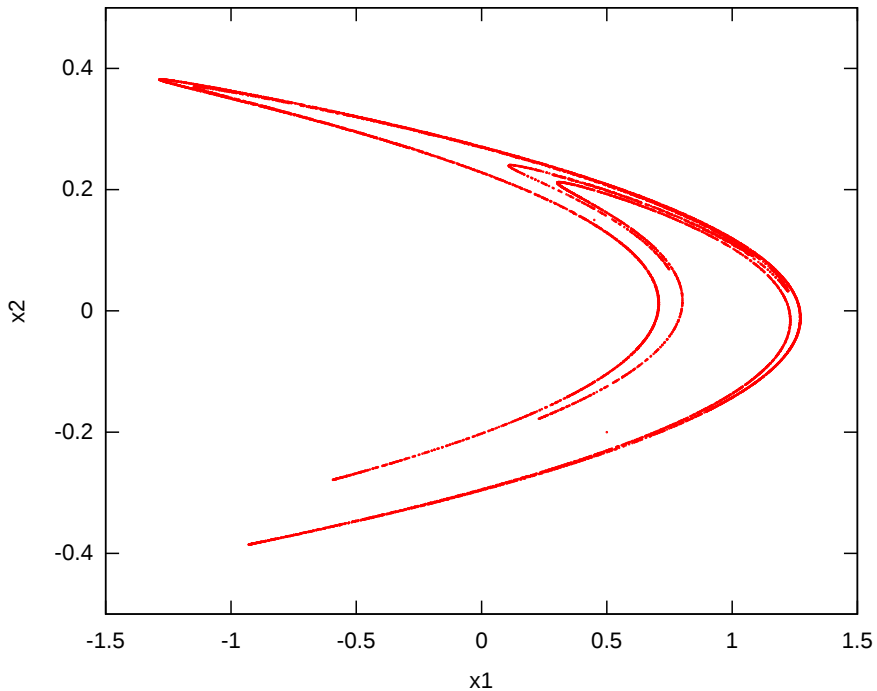

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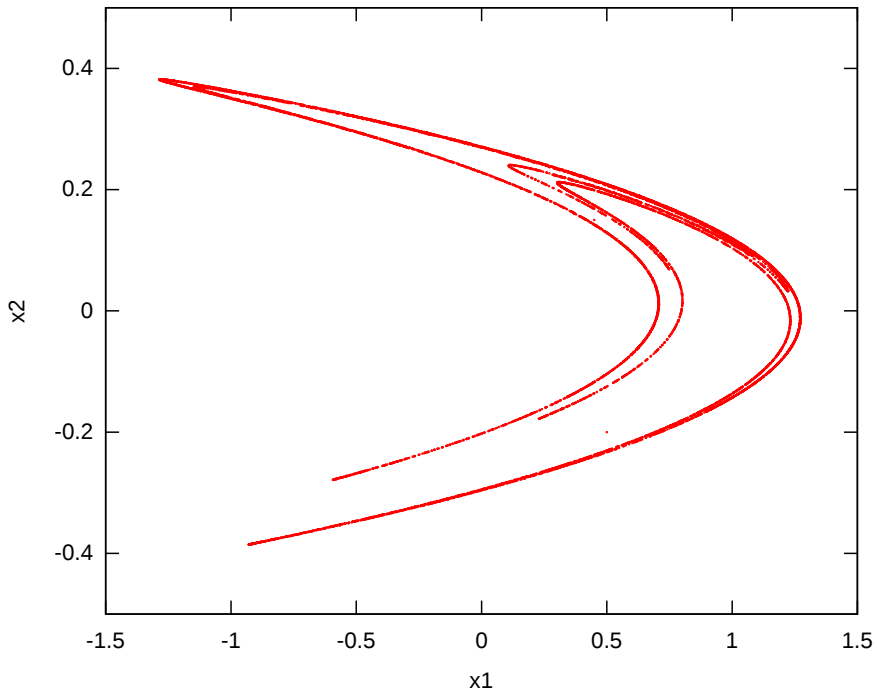

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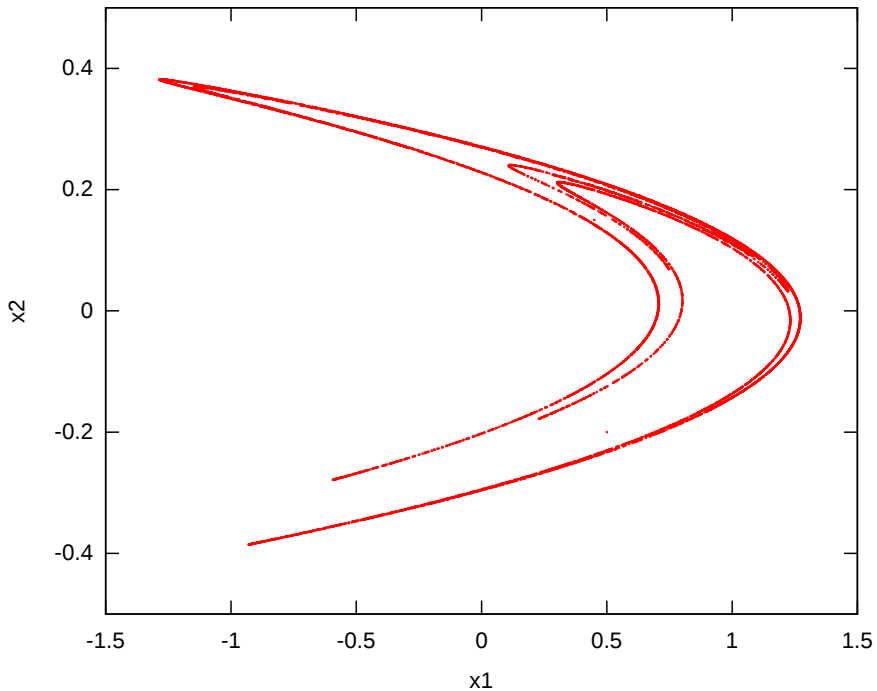

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- there is a set of parameters (near $b = 0$) with positive Lebesgue measure for which the Hénon map has a strange attractor. [Benedicks & Carleson,'91].

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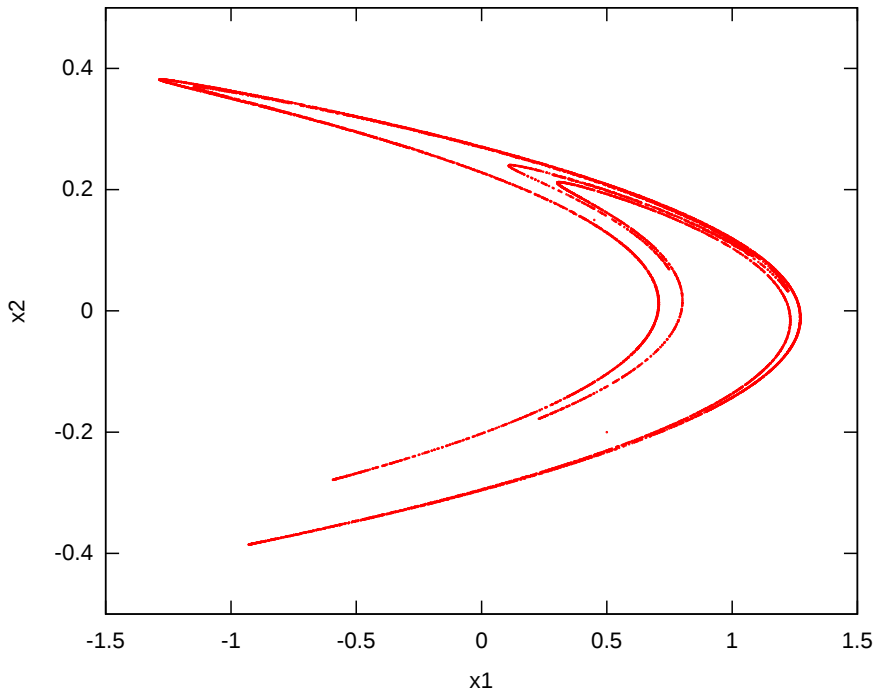

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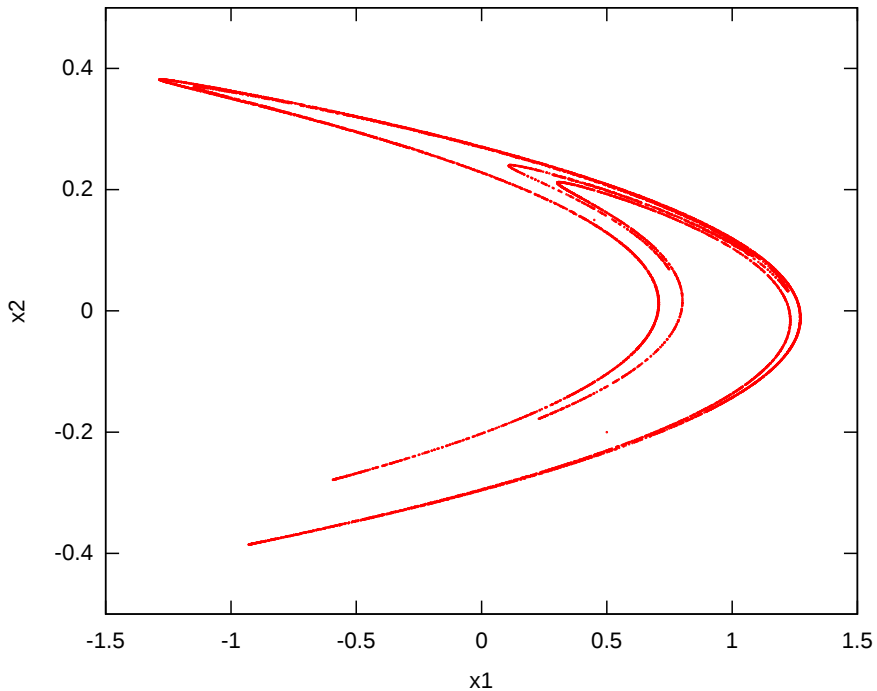

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Example of sink in Hénon attractor [Galias & Tucker 2013]

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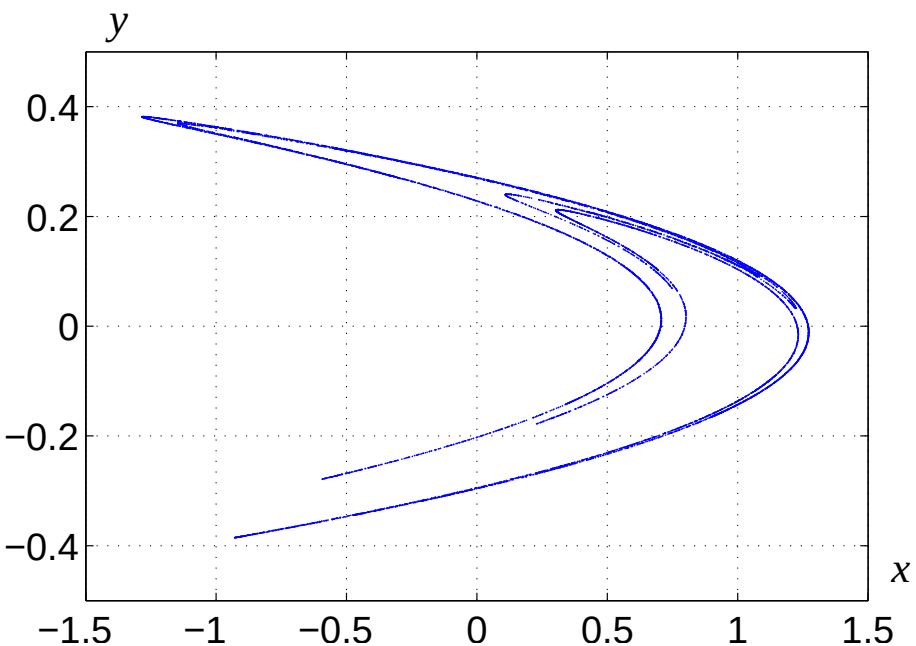
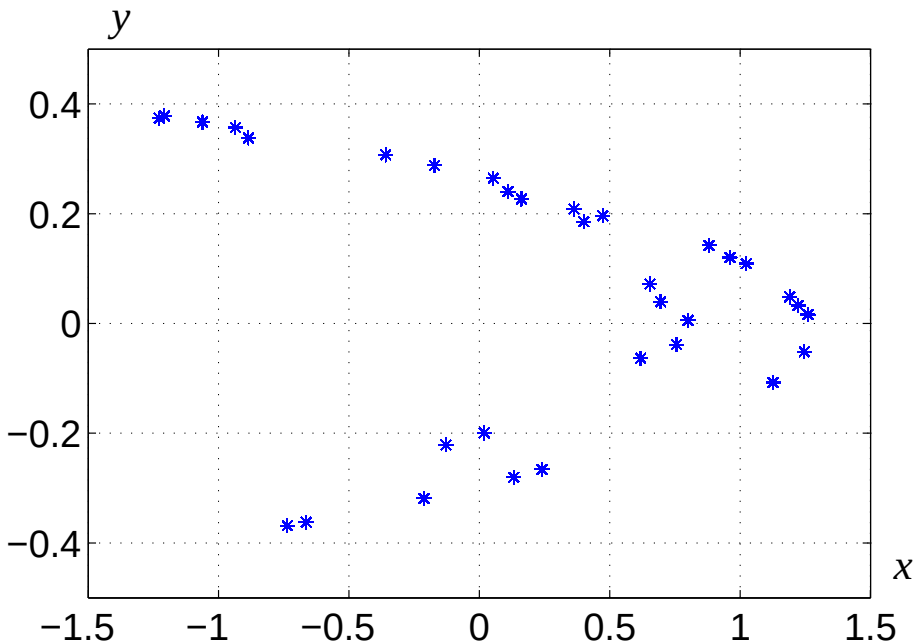
Let $a = 1.399999486944$, $b = 0.3$. Trajectory composed of 10000 points:



(a) $5 \cdot 10^9$ iterations skipped



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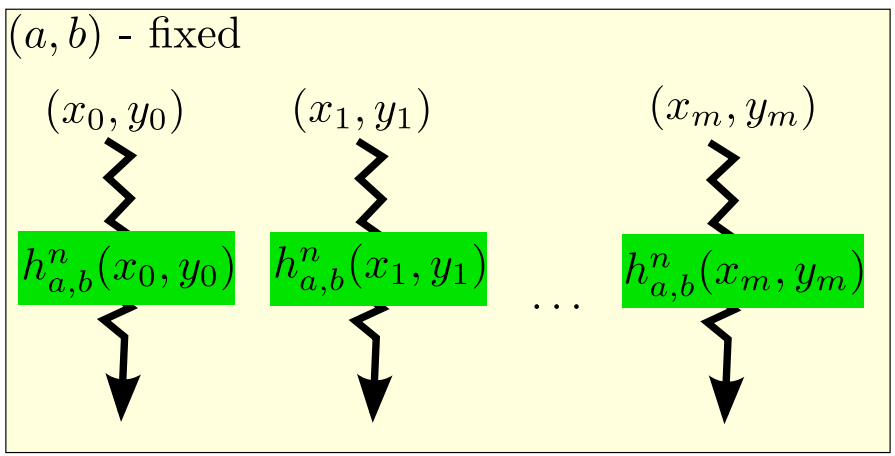

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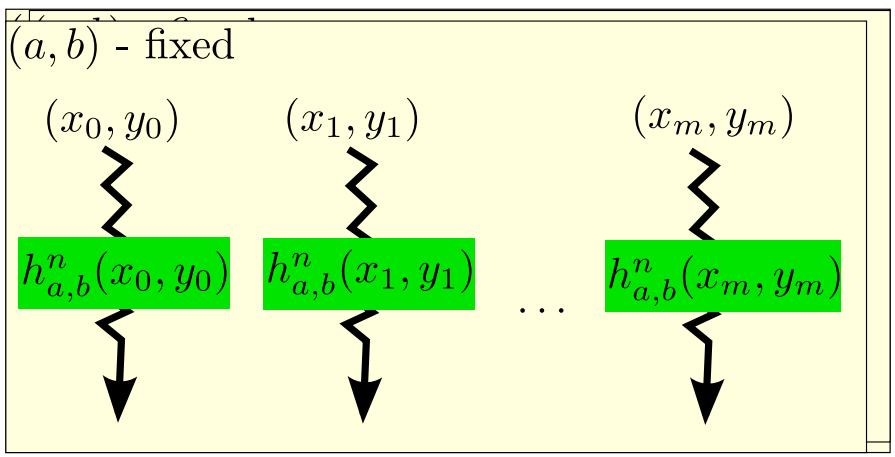

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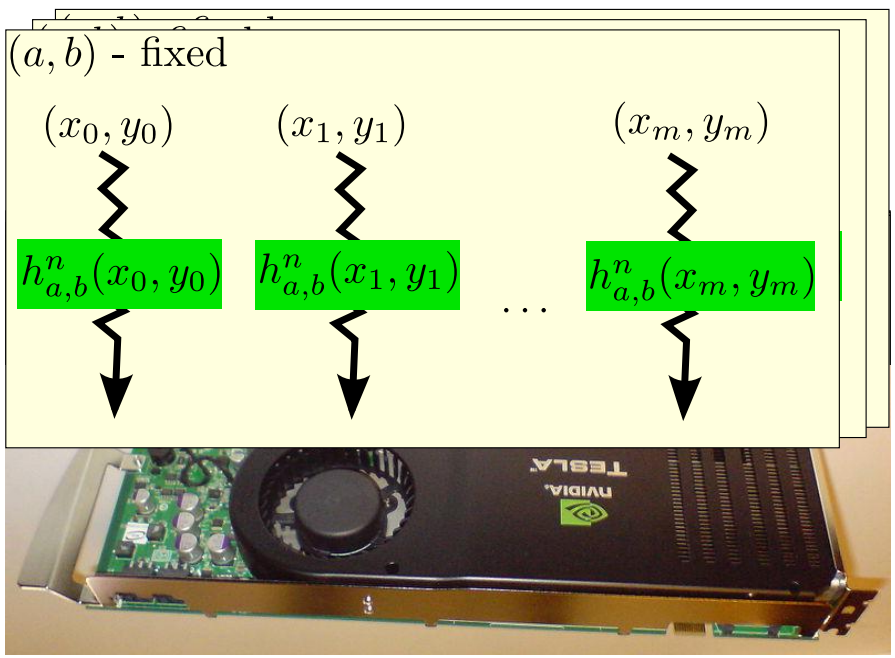
Graphics Processing Units
(GPUs)



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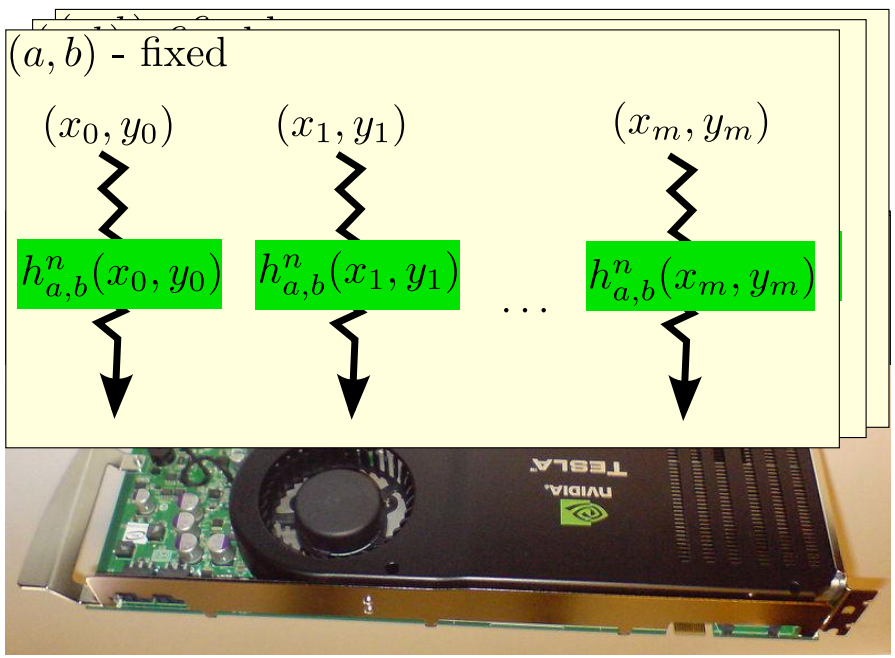
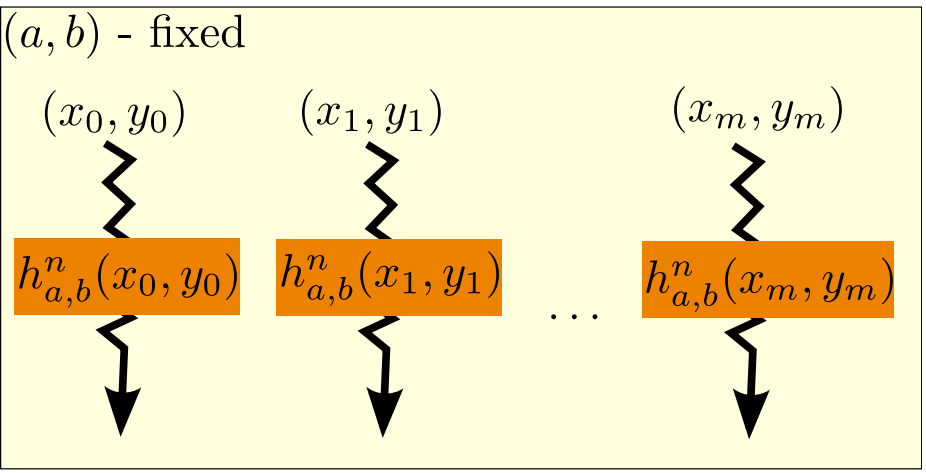
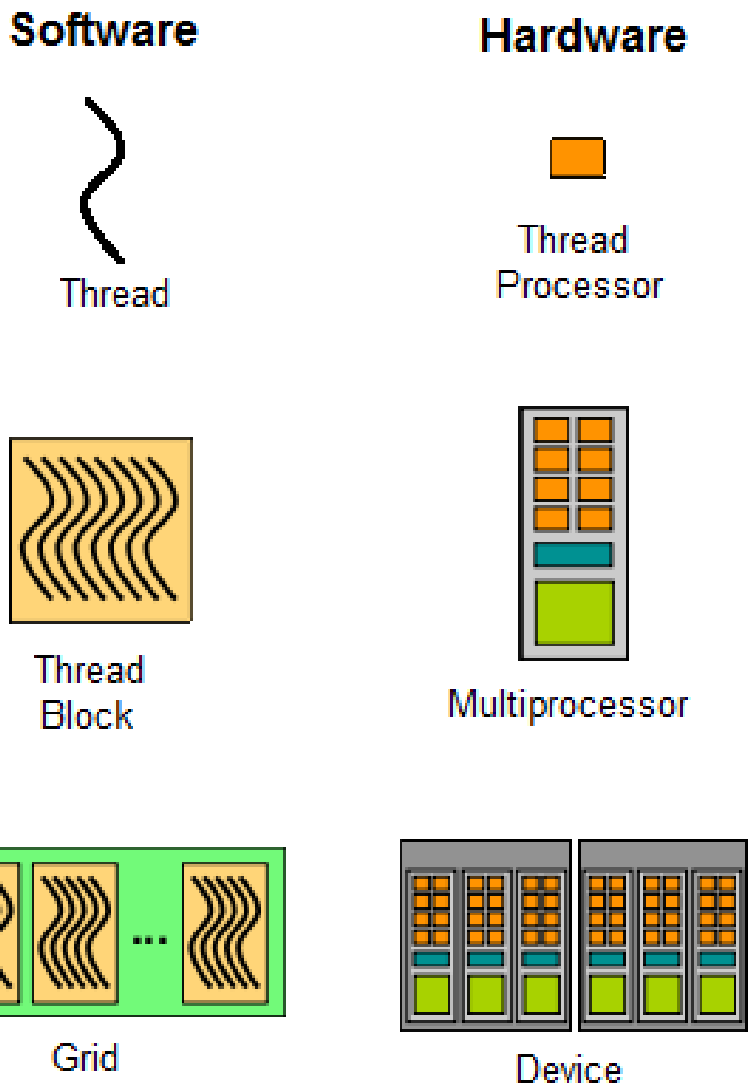


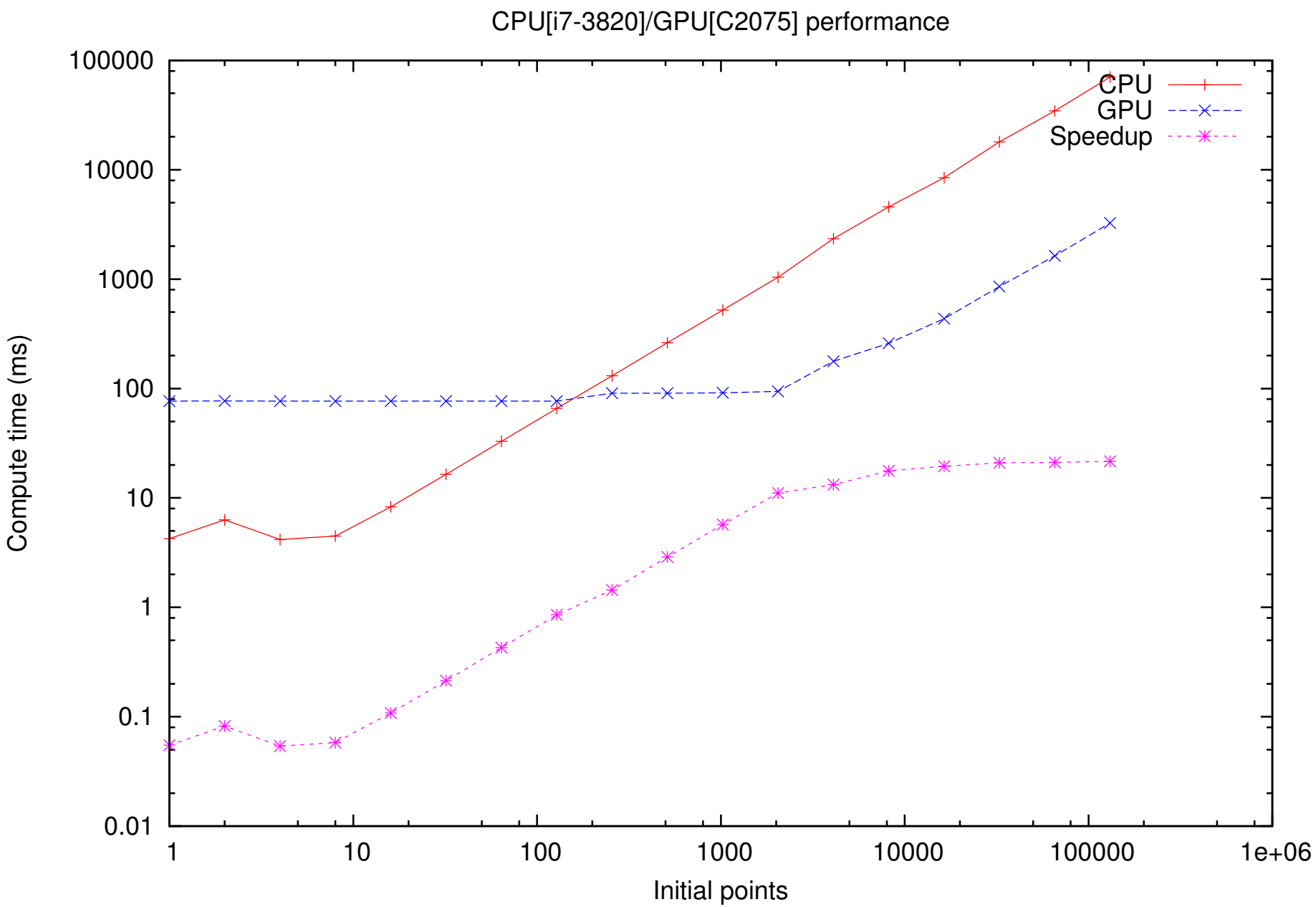
Hénon on GPU - Execution model in a nutshell





Example: Fermi Arch – up to 512 thread processors in 16 multiprocessors

Hénon on GPU - Speed-up



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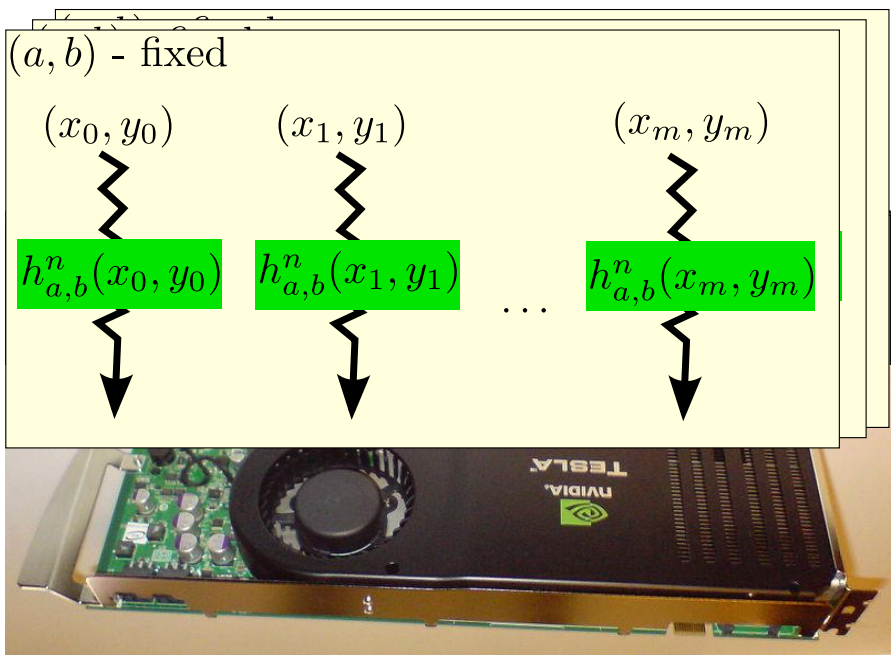

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2. A posteriori validation of existence and stability with interval arithmetic

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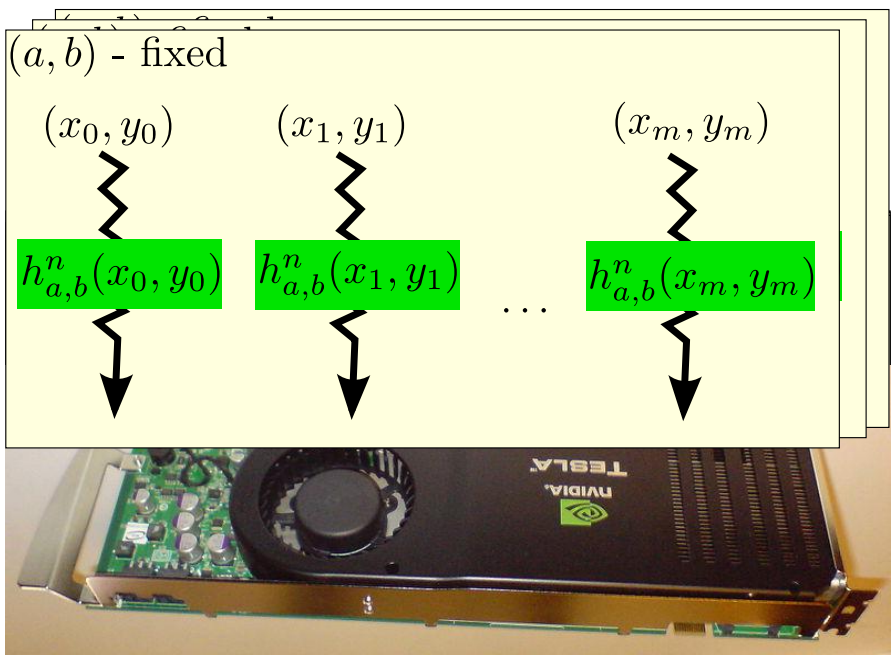

Rigorous a posteriori proof of existence: interval Newton operator [Neumaier 1990]

Theorem

Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, $F \in \mathcal{C}^1(D)$, $\boldsymbol{v} \in \mathbb{ID}$, $\hat{v} \in \boldsymbol{v}$.

$$N(\boldsymbol{v}) := \hat{v} - F'(\boldsymbol{v})^{-1} F(\hat{v})$$

If $N(\boldsymbol{v})$ is well defined, then the following statements hold:

(1) if $\boldsymbol{v}$ contains a zero x^* of F, then so does $N(\boldsymbol{v}) \cap \boldsymbol{v}$;

(2) if $N(\boldsymbol{v})) \cap \boldsymbol{v} = \emptyset$, then $\boldsymbol{v}$ has no zeros of F;

(3) if $N(\boldsymbol{v})) \subseteq \boldsymbol{v}$, then $\boldsymbol{v}$ contains a unique zero of F;

Proof.

(1) Follows from Mean Value Theorem;

(2) Contra-positive of (1);

(3) Existence from Brouwer's fixed point theorem; uniqueness from non-vanishing F'.

□

$\mathbb{ID}$ is the set of all real intervals included in D.

Validation of periodic orbits using Newton operator [Galias 2001], [Galias & Tucker, 2011, 2013]

- Let $v = (z_0, z_1, \ldots, z_{p-1})$ be the computer generated period p orbit.

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Remark

Let $v = (z_0, z_1, \ldots, z_{p-1})$ and

$$[F(v)]_k = z_{(k+1) \bmod p} - h(z_k), k = 0, \ldots, p-1.$$

Then $F(v) = 0$ iif $h^p(z_0) = z_0$.

$\longrightarrow$ existence of zeros of $\mathrm{Id} - h^p \iff$ existence of zeros of F;
$\longrightarrow$ reduced overestimation using F [Galias 2001].

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- Verify that $N(\boldsymbol{v}) \subseteq \boldsymbol{v}$.

Floating-Point Multiple Precision on Graphics Processing Units (GPUs)

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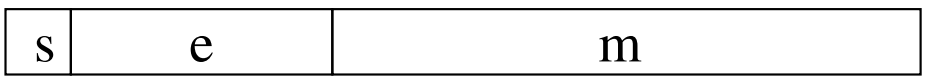
1. Floating Point (FP): A real number X is approximated in a machine by a rational $x = (-1)^s \times m \times \beta^e$
   - β is the radix (usually 2)
   - s is the sign bit
   - m is the mantissa, a rational number of p digits in radix β

     $$m = d_0.d_1 d_2 \dots d_{p-1}$$

   - e is the exponent, a signed integer on n_e bits



High parallelism + Multiple precision → Need for Multiple Precison on GPUs

1. Floating Point (FP): A real number X is approximated in a machine by a rational $x = (-1)^s \times m \times \beta^e$
   - β is the radix (usually 2)
   - s is the sign bit
   - m is the mantissa, a rational number of p digits in radix β

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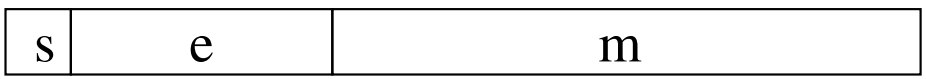


2. GPUs:
   - used for scientific computing: GP-GPU programs
   - Languages: OpenCL, CUDA (NVIDIA)
   - conformant with IEEE-754 standard
   - Support for all rounding modes for basic operations $(+, -, *, /, \sqrt{})$
   - Dynamic rounding mode without any penalties:

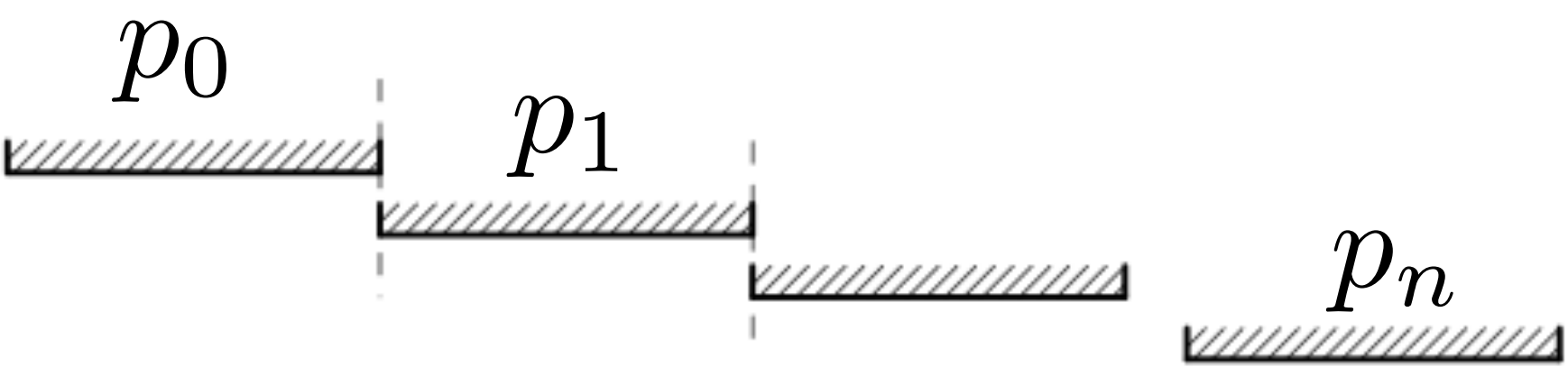
     `__dadd_rn(a,b)=RN(a+b)`
   - The FMA (Fused Multiply-Add):

     `__dfma_rn (a,b,c)= RN(A *B + C)`

Our approach: multiple-term expansions, i.e.

a number is expressed as a non-evaluated sum of FP numbers
$N = p_0 + \ldots + p_n$.

The expansions are required to be non-overlapping.



Previous works: Zimmer, et al. (C - XSC Library with Kulisch long accumulator), Dekker71, Priest91, Shewchuk96, Bailey01, Nievergelt04, Rump05

Precision-2p vs. double-double

π in double-double

$$p_1 = 11.0010010000111111011010101000100010000101101000110000_2,$$

and

$$p_2 = 1.0001101001100010011000110011000101000101110000000111_2 \times 2^{-53}.$$

$p_1 + p_2 \leftrightarrow 107$ bits FP approx.

Non-overlapping FP Sequences

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$\ln(12)$ in double-double

$$\ell_1 = 10.011111000010001011010111100110100111001111001111101_2,$$

and

$$\ell_2 = -1.100110001110010000001111100001011011110101111001011 1_2 \times 2^{-55}.$$

$\ell_1 + \ell_2 \leftrightarrow 109$ bits FP approx.

Error-Free Transformations: 2Sum & 2Prod

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Example 2Sum:

$$
\begin{aligned}
a &= 1.\underbrace{0\ldots0}_{50}11 \times 2^0 \\
b &= 0.\underbrace{0\ldots00}_{51}111 \times 2^0 = 1.11 \times 2^{-52}
\end{aligned}
$$

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s &= 1.\underbrace{0\ldots0}_{49}101 \times 2^0 \\
t &= -0.\underbrace{0\ldots00}_{53}1 \times 2^0 = -1 \times 2^{-54} \\
a + b = s + t &= 1.\underbrace{0\ldots0}_{49}10011 \times 2^0
\end{aligned}
$$

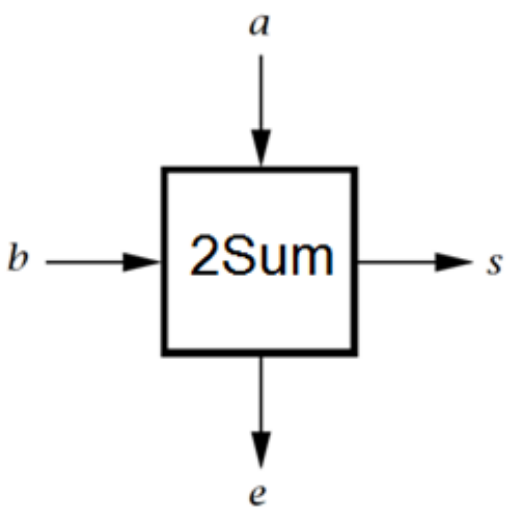
Error-Free Transformations: 2Sum & 2Prod

Assume the floating-point system being used has radix $\beta \leq 3$, subnormal numbers available, and provides correct rounding with rounding to nearest.

Theorem 1 (2Sum algorithm)

Let a and b be floating-point numbers. Algorithm 2Sum computes two floating-point numbers s and t that satisfy the following:

- $s + t = a + b$ *exactly;*
- $s = RN(a + b)$.



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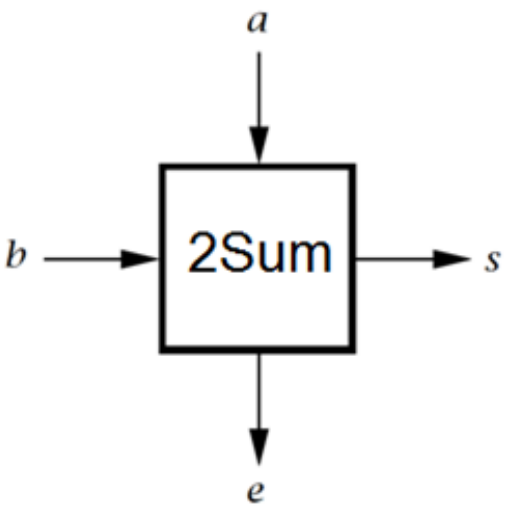



```
/* Computes RN(a+b) and t=(a+b)-RN(a+b).  */
__device__ static __forceinline__
  double 2Sum(double a, double b, double &t) {
  double s = __dadd_rn(a,b);
  double aa = __dadd_rn(s,-b);
  double bb = __dadd_rn(s,-aa);
  double da = __dadd_rn(a,-aa);
  double db = __dadd_rn(b,-bb);
  t= __dadd_rn(da,db);
  return s;
}
```

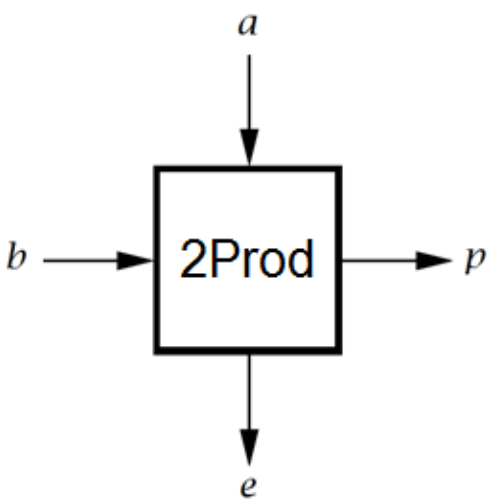
Error-Free Transformations: 2Sum & 2Prod

Assume the floating-point system being used has radix $\beta \leq 3$, subnormal numbers available, and provides correct rounding with rounding to nearest.

Theorem 2 (2ProdFMA algorithm)

Let a and b be floating-point numbers, $e_a + e_b \geq e_{min} + p - 1$. Algorithm 2ProdFMA computes two floating-point numbers p and e that satisfy the following:

- $p + e = a * b$ *exactly;*
- $p =$ *RN*$(a * b)$.



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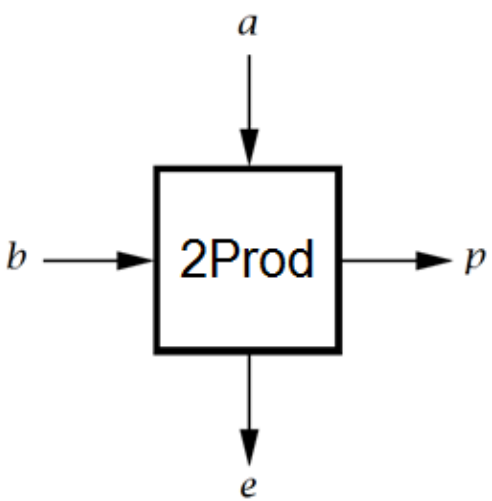
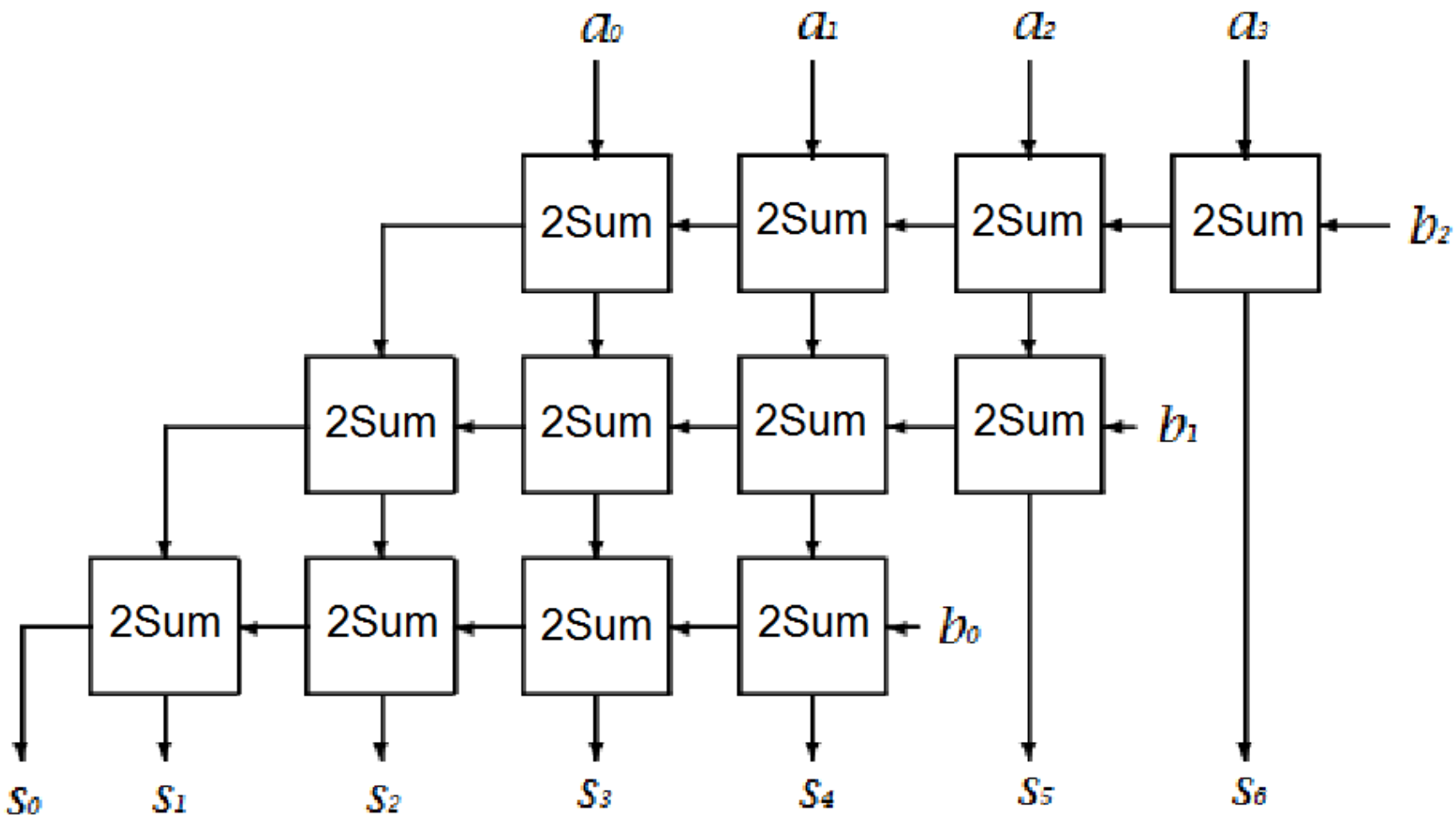



```
/* Computes p=RN(a*b) and e=(a*b)-p, using an FMA.
__device__ static __forceinline__
  double 2ProdFMA(double a, double b, double &e) {
  double p = __dmul_rn(a, b);
  e = __fma_rn(a, b, -p);
  return p;
}
```

Shewchuck's ExpansionSum

Input: (a_i), (b_i) non-overlapping expansions

Output:(s_i) non-overlapping expansion, s.t. $\sum_{i=1}^{n+m} s_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{m} b_i$



Parallelisation of search for sinks on Haddock cluster

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  $n \sim 10^6 - 10^9$.
- Code example:

```
#define prec 4   /*kernel to be run using quad prec*/
__device__ void henon_iterate(double x0, double y0,
                  double a, double b, long int ITER) {
    /*init multi_prec template vars*/
    multi_prec<prec,double> x_i(x0);
    multi_prec<prec,double> y_i(y0);
    multi_prec<prec,double> x_old;
    for (long int i=1; i <= ITER; i++) {
        /*Compute iterates*/
        x_i = y_i + 1.0 - a*x_i*x_i;
        y_i = b*x_old;
    }
}
```

Found Hénon sinks [Galias & Tucker, 2013]

Closest to classical parameters, so far: $a = 1.399999486944, b = 0.3$.

a	p	d	r	λ_1
1.399922051	25	$5.522-12$	$2.473-12$	-0.00132
1.39997174948	30	$1.354-11$	$3.561-12$	-0.01887
1.3999769102	18	$3.207-09$	$1.014-09$	-0.05306
1.39998083519	24	$1.703-11$	$7.384-12$	-0.02819
1.399984477	20	$8.875-10$	$4.076-10$	-0.05099
1.39999492185	22	$3.686-11$	$1.531-11$	-0.09600
1.3999964733062	39	$2.784-13$	$1.115-13$	-0.03547
1.399999486944	33	$1.110-12$	$6.901-13$	-0.01843
1.40000929916	25	$1.118-11$	$5.128-12$	-0.08379
1.4000227433	21	$2.262-10$	$7.901-11$	-0.05612
1.40002931695	27	$5.782-11$	$2.646-11$	-0.01140
1.40006377472	27	$8.692-11$	$3.810-11$	-0.05636
1.40006667358	24	$6.278-11$	$2.646-11$	-0.01112
1.4000843045	27	$9.400-11$	$4.572-11$	-0.06870
1.40009110518	22	$3.493-11$	$1.531-11$	-0.02157
1.4000967515	26	$2.463-10$	$1.365-10$	-0.13233

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2 doubles	1798
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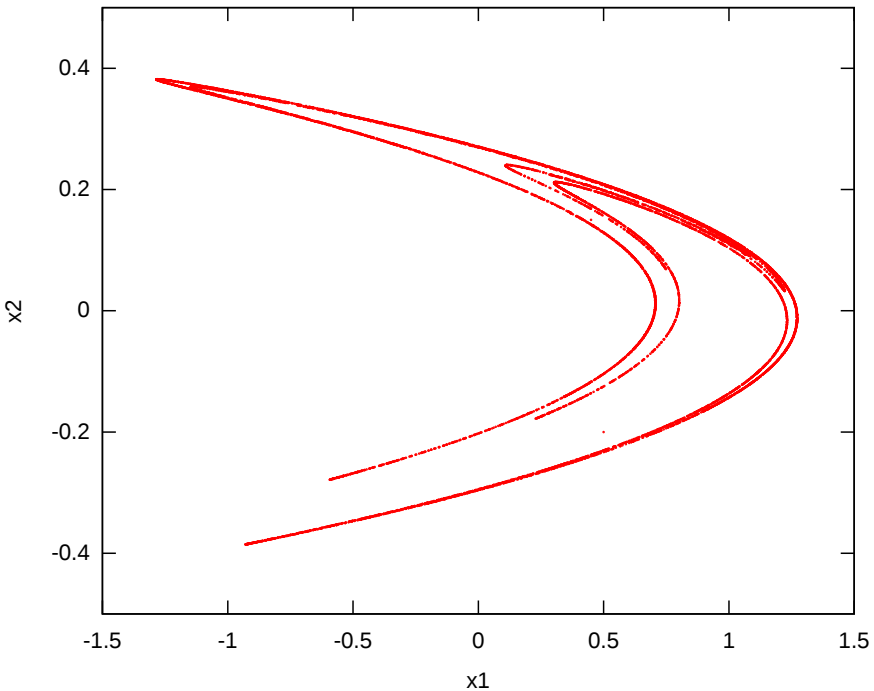
- Correct rounding for operations with FP expansions ?
- Iterate other dynamical systems ?

Conclusion

Strange Attractors



Quoting Ruelle...

I asked Floris Takens if he had created this remarkably successful expression. Here is his answer: "Did you ever ask God whether he created this damned Universe? ... I don't remember anything ... I often create without remembering it..." The creation of strange attractors thus seems to be surrounded by clouds and thunder. Anyway, the name is beautiful, and well suited to these astonishing objects, of which we understand so little.

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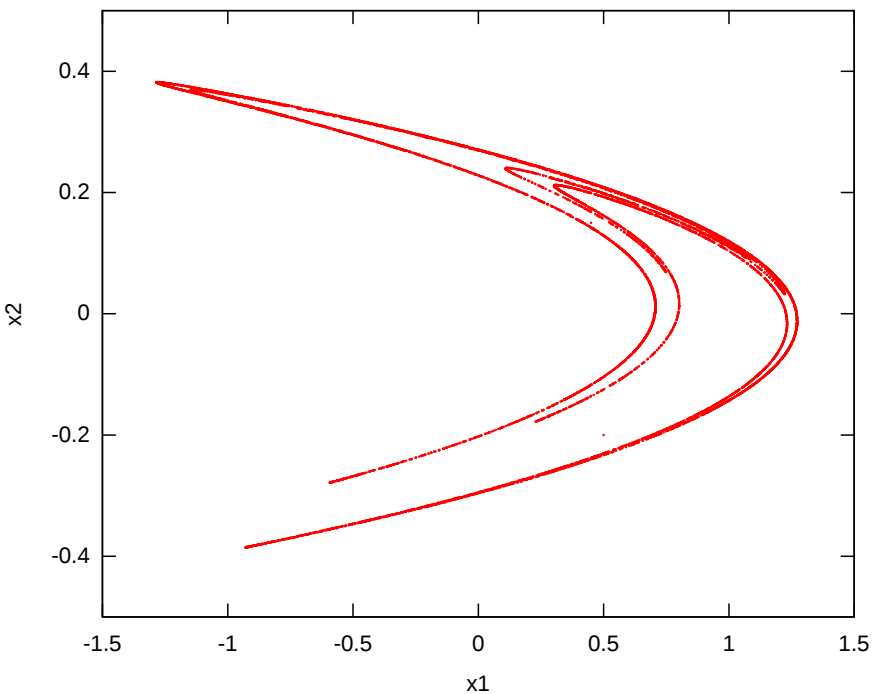

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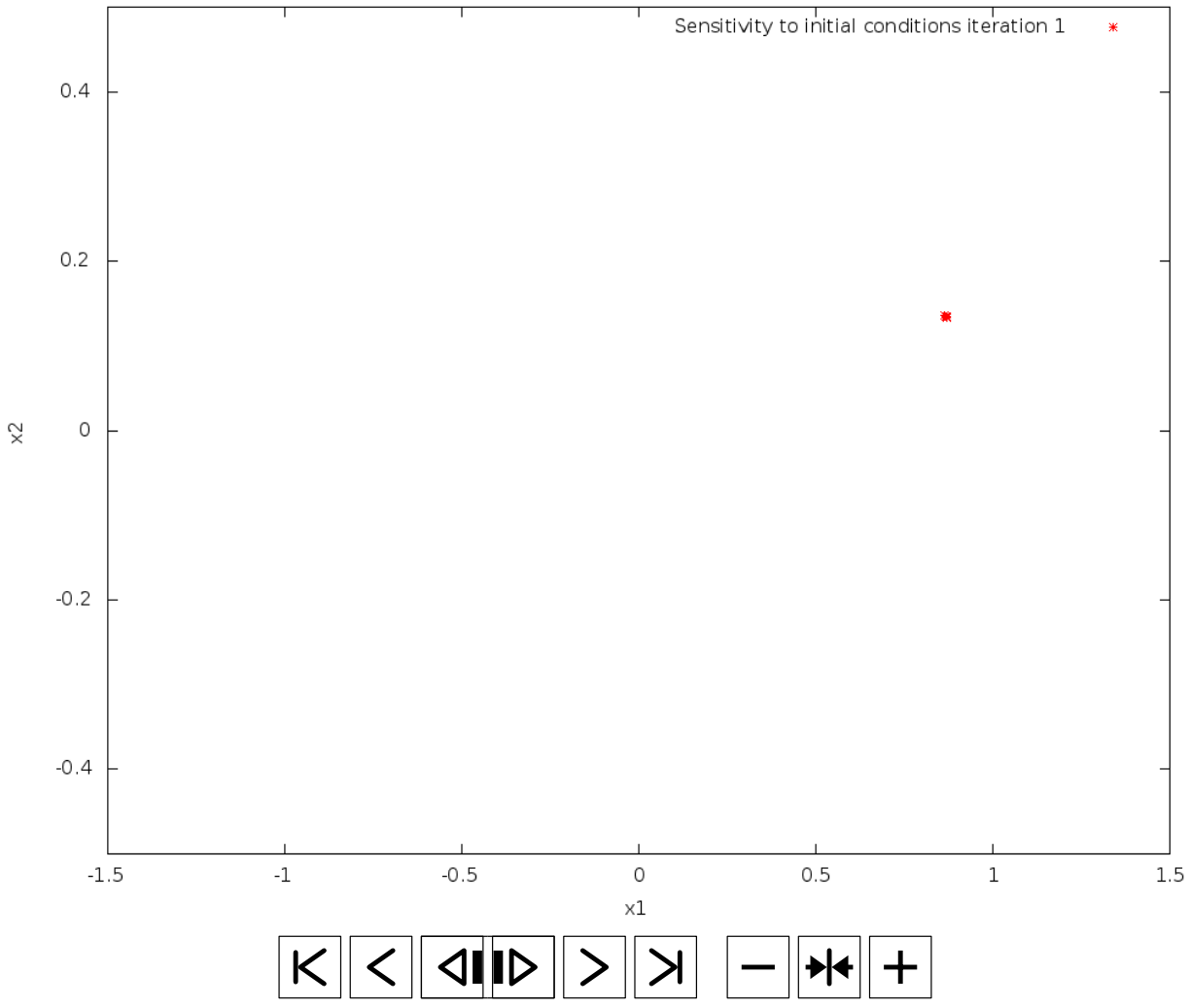
Hénon Attractor - Sensitivity to initial conditions

Hénon Map

$$h_{a,b}(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1)$$



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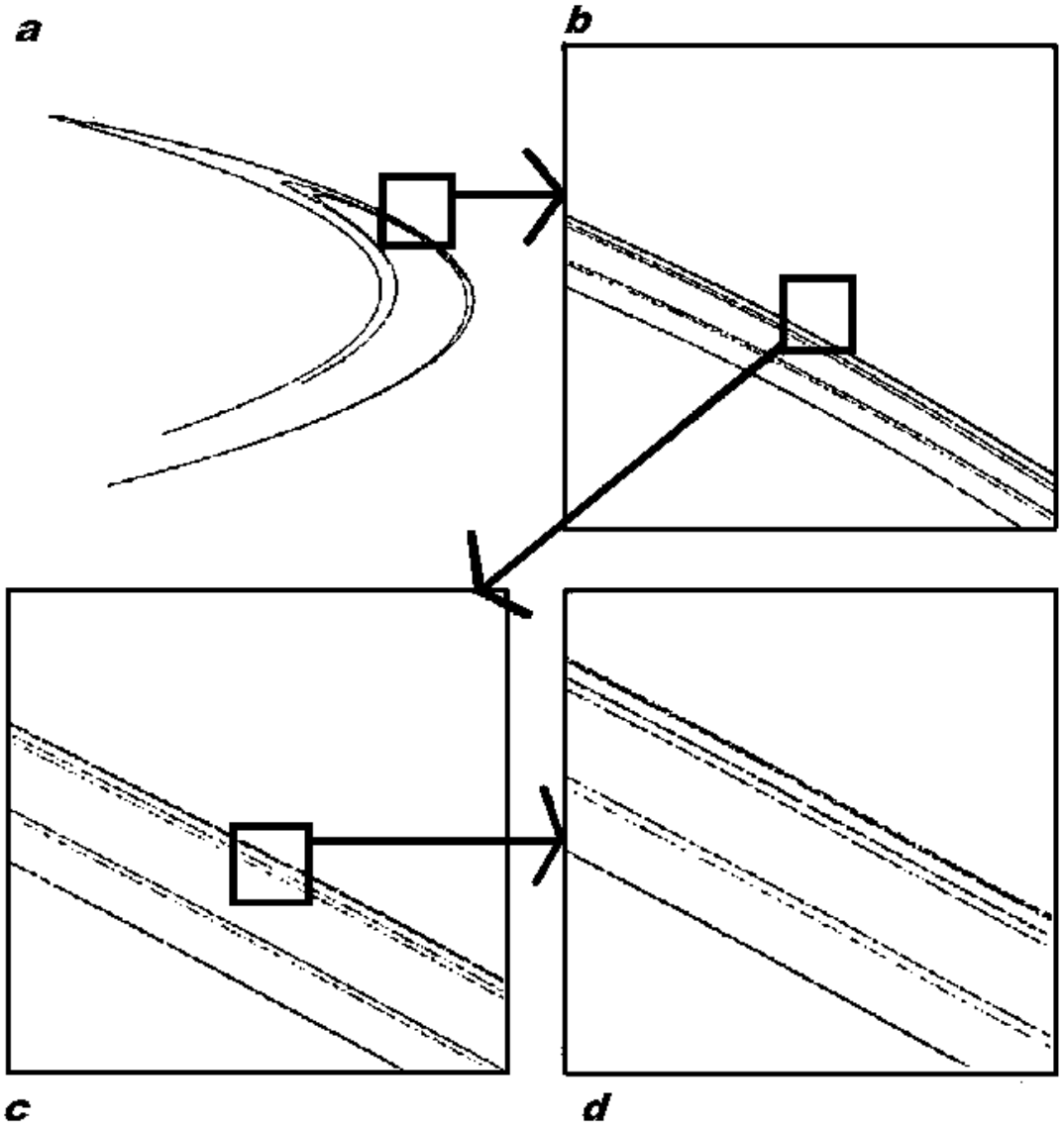
Hénon Attractor - Fractal Dimension



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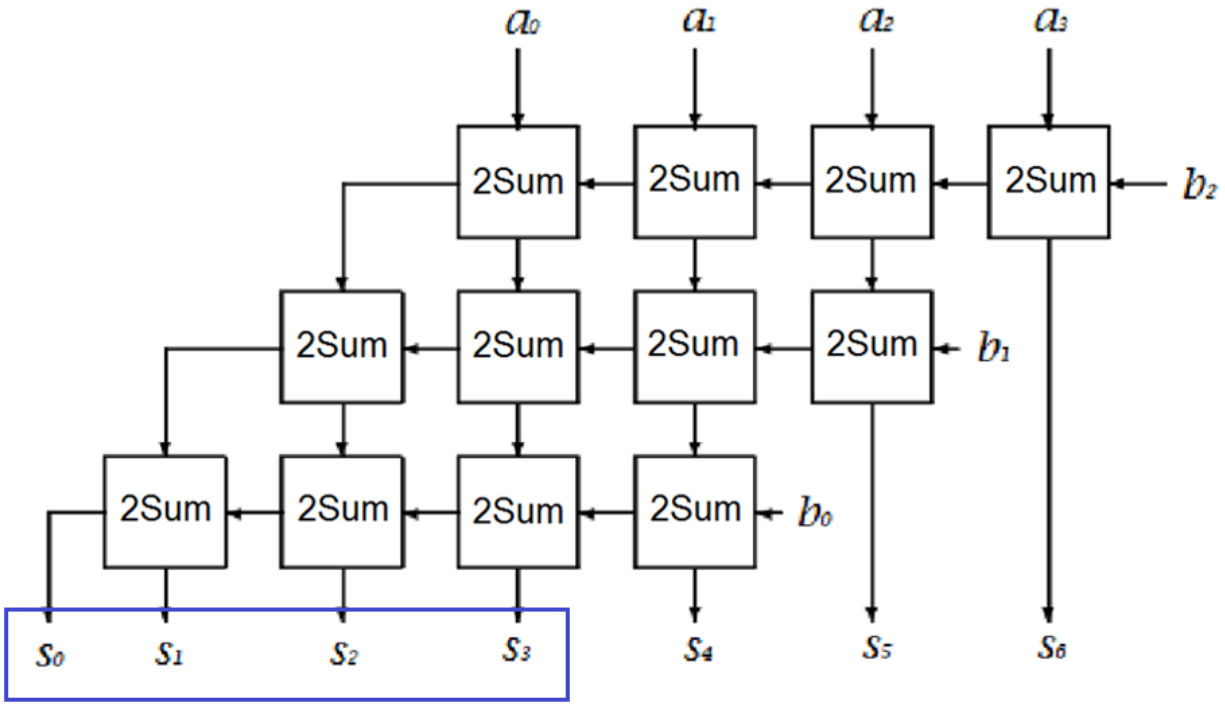
Shewchuck's ExpansionSum



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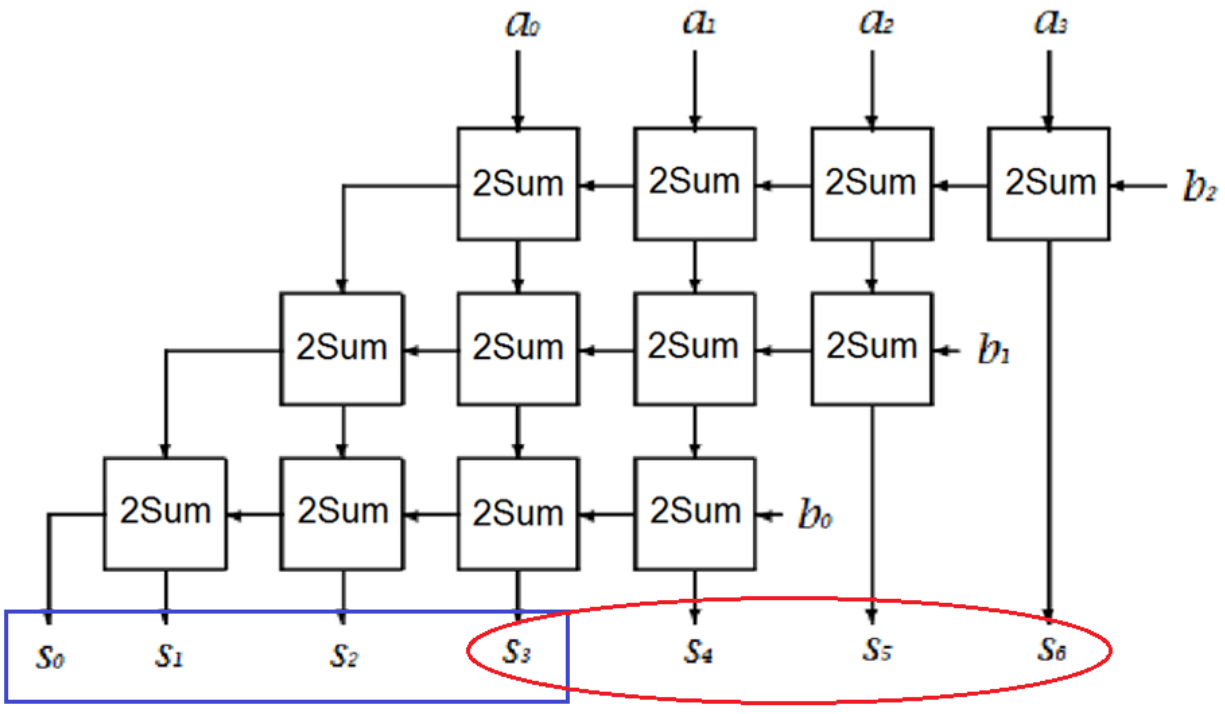

$$u = \mathsf{RU}\left(\sum_{i=r-1}^{n+m-1} s_i\right); \qquad flSum = \mathsf{RN}\left((u+d)/2\right);$$

$$d = \mathsf{RD}\left(\sum_{i=r-1}^{n+m-1} s_i\right); \qquad \varepsilon = \max\left(\mathsf{RU}\left(flSum - d\right), \mathsf{RU}\left(u - flSum\right)\right);$$

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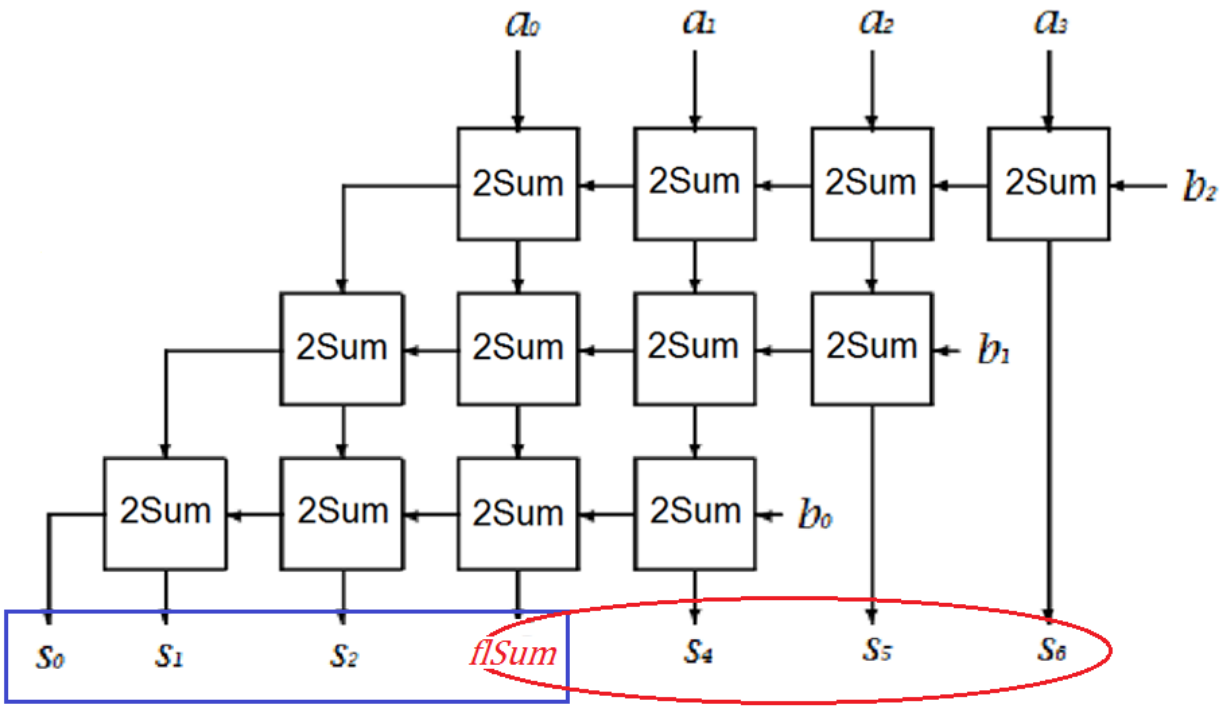

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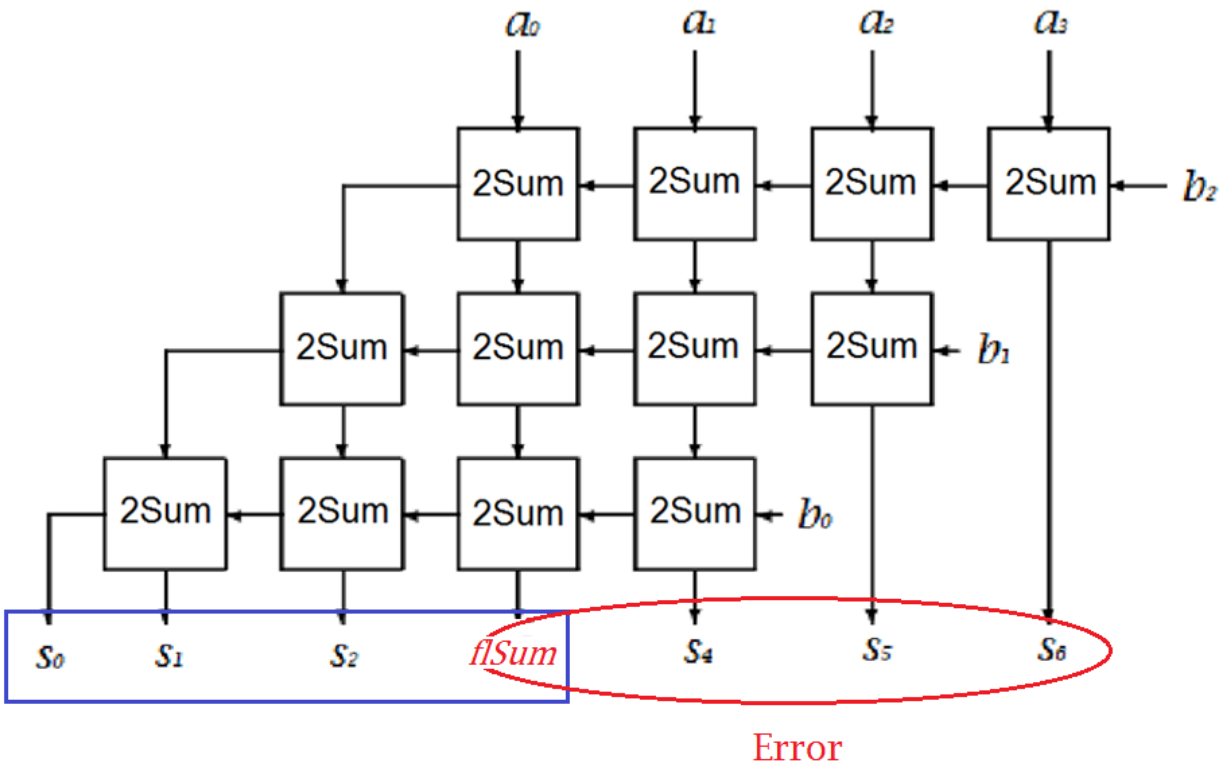

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Interval arithmetic support: $(\sum_{i=1}^{r} s_i, \varepsilon)$ with very little overhead.