6ème Rencontres Arithmétiques de l'Informatique Mathématique

RAIM2013

Session Arithmétique et Musique 18.11.2013

Sturmian involution, lattice-path-transformations and their application to the investigation of diatonicity

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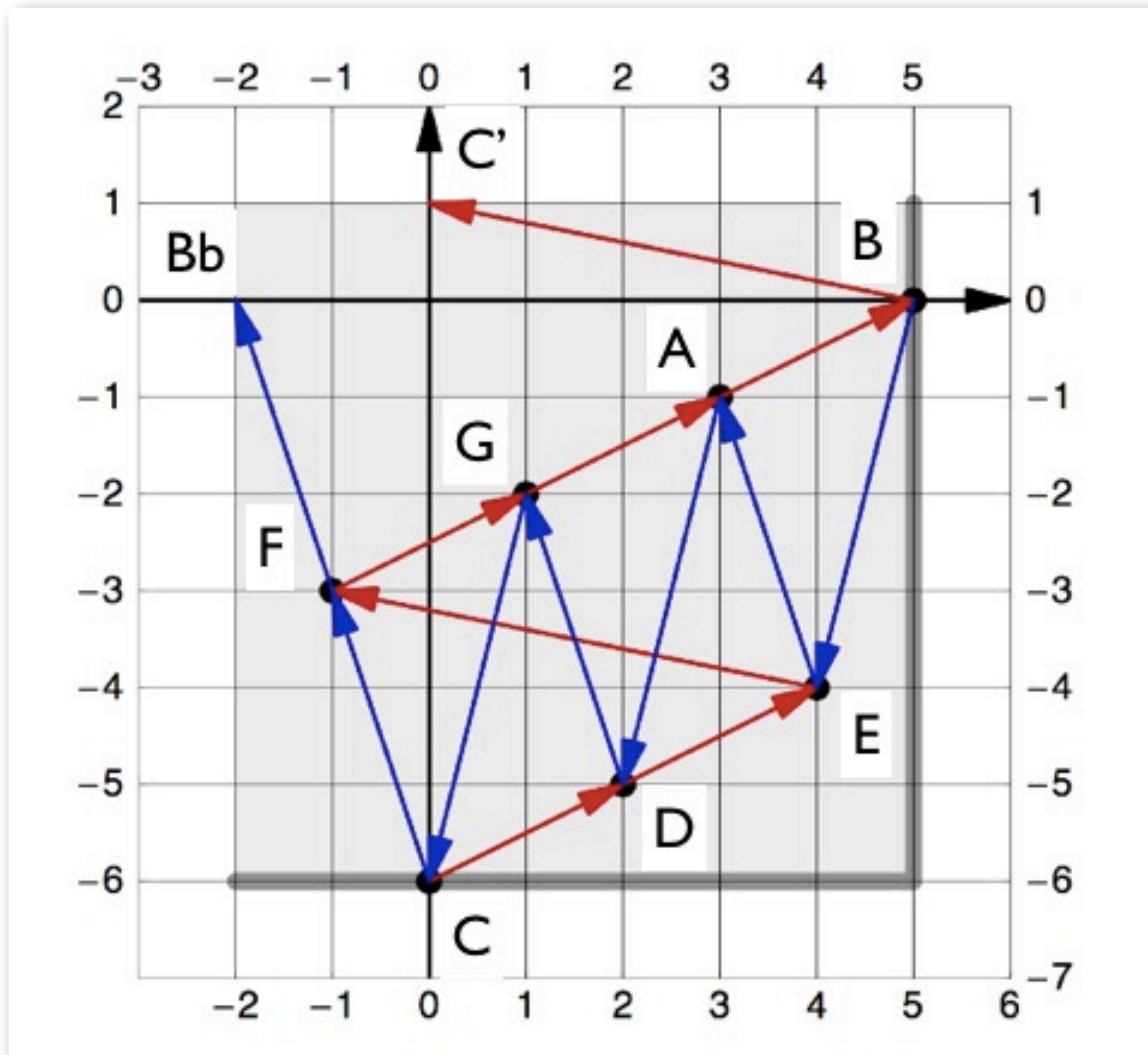
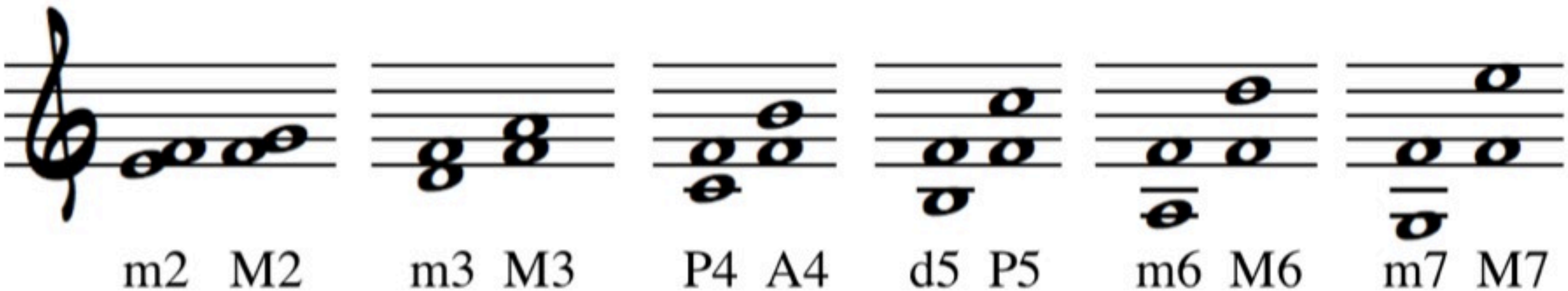
esmuc

ESCOLA SUPERIOR DE MÚSICA DE CATALUNYA



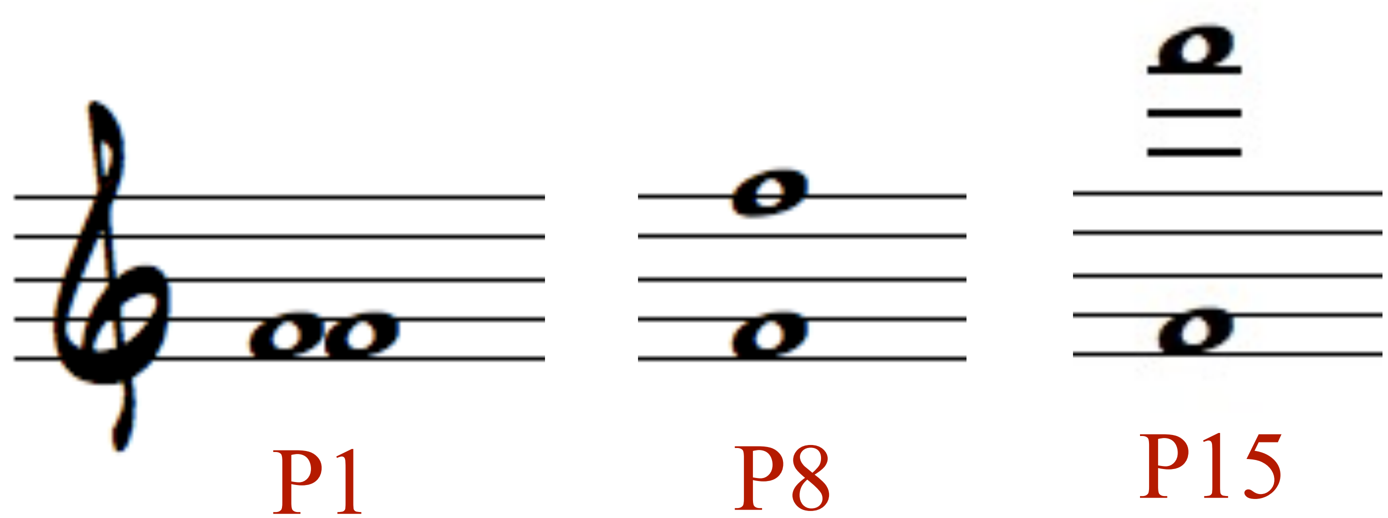
Elementary Observations about Musical Notation and the Diatonic System



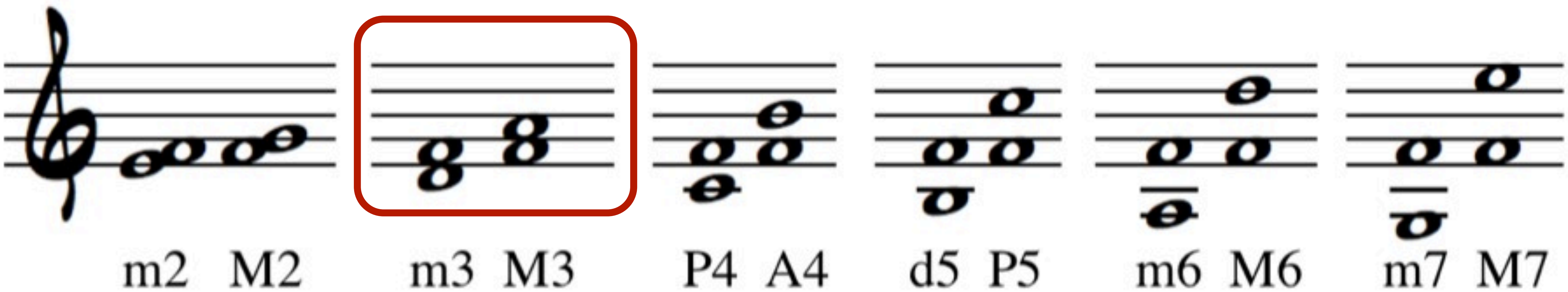
(Almost) every generic interval comes in two species.

"Myhill's Property"

Exceptions:



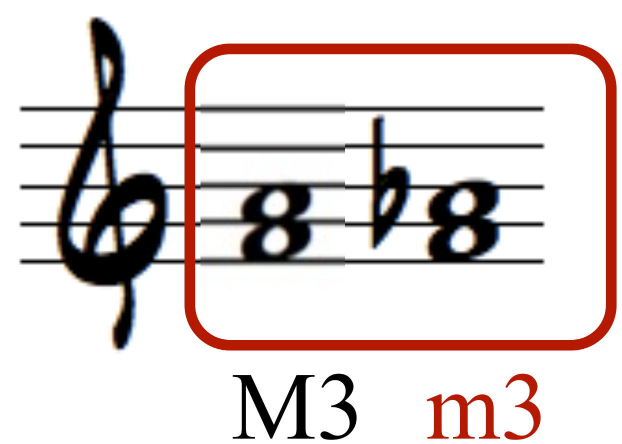
Elementary Observations about Musical Notation and the Diatonic System



(Almost) every generic interval comes in two species.

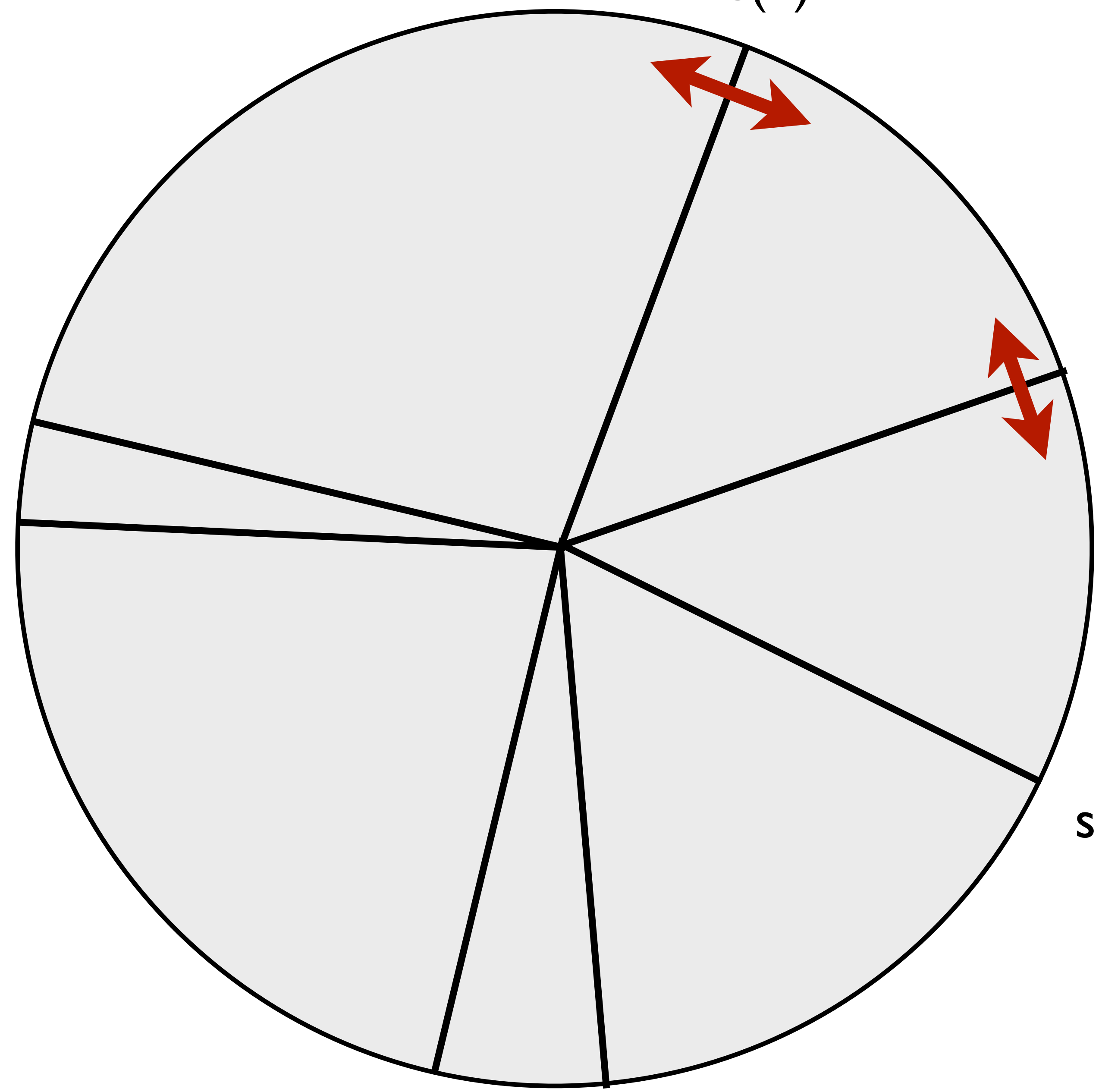
"Myhill's Property"

The two species always differ by an alteration (# or b):



An "Arbitrary" Scale

$$s : \mathbb{Z}_n \xrightarrow{\sim} S \subset \mathbb{R}/\mathbb{Z}$$



represented in ascending ordering with: $s(0) < s(1) < .... < s(n-1) < s(0) + 1$

The Well-formedness Property

Scale:

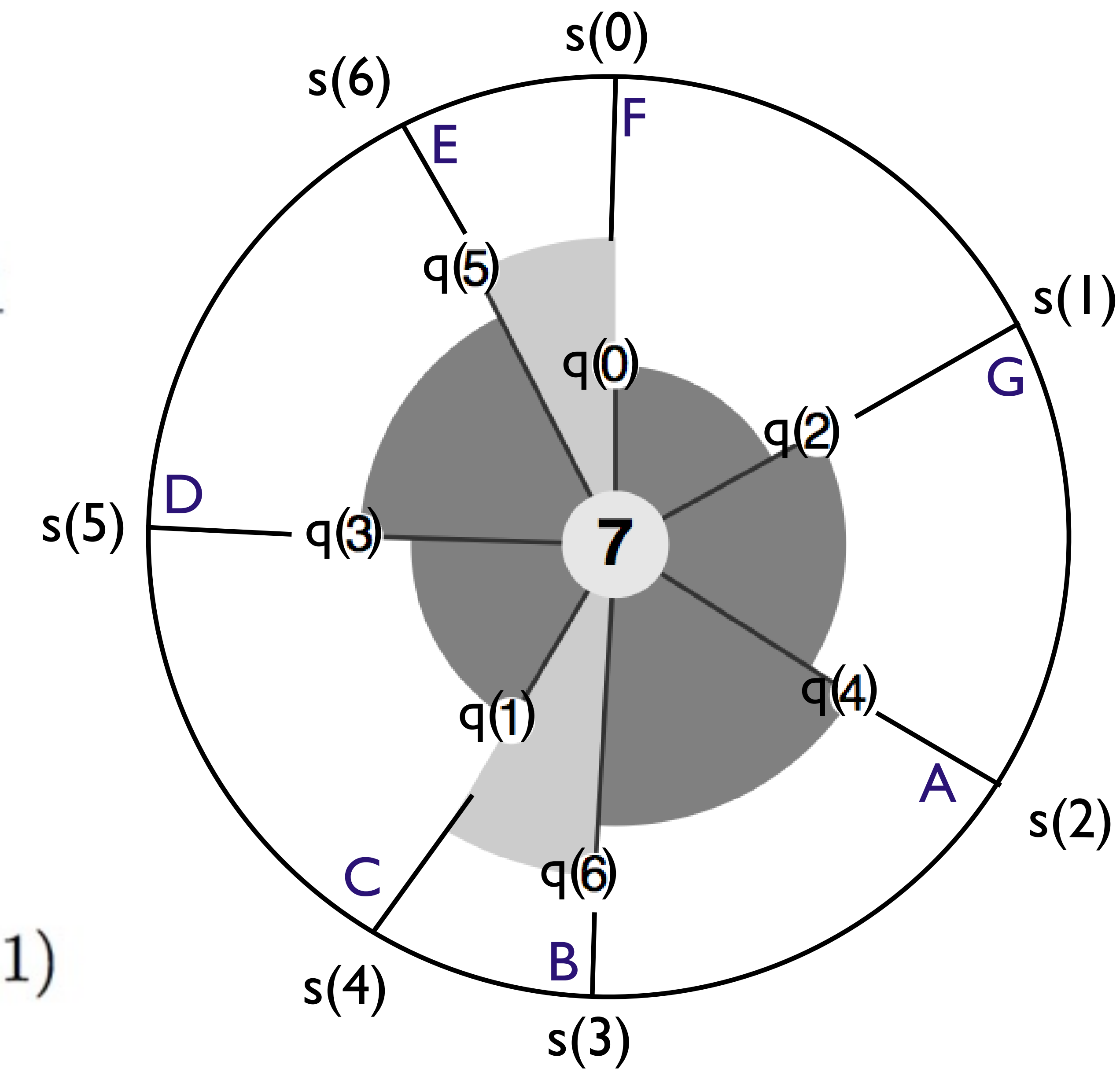
$$s : \mathbb{Z}_n \xrightarrow{\sim} S \subset \mathbb{R}/\mathbb{Z}$$

$$s(0) < s(1) < .... < s(n-1) < s(0) + 1$$

Generated Scale:

$$\begin{array}{ccc} \mathbb{Z}_n & & \\ p \downarrow \wr & \searrow^{q}_{\wr} & \\ \mathbb{Z}_n & \xrightarrow[s]{\sim} & S \subset \mathbb{R}/\mathbb{Z} \end{array}$$

$$q(k) = g \cdot k \ mod\ 1 \text{ for some } g \in (0,1)$$



Well-formed Scale:

The permutation $p = s^{-1}q : \mathbb{Z}_n \tilde{\to} \mathbb{Z}_n$ is a linear map.

Theorem (Carey & Clampitt 1989, 1996):

Consider an n-tone scale $S = \{k \cdot g \bmod 1 \mid k = 0, ..., n-1\} \subset \mathbb{R}/\mathbb{Z}$, generated by $g \in (0,1)$. Let $q, s : \mathbb{Z}_n \to S$ denote the associated generation order and scalar order encodings of S, respectively. The following three properties are equivalent:

(i) S is non-degenerate well-formed, i.e. $s^{-1}(q(k)) = m \cdot k \bmod n$ for a suitable $m \in \mathbb{Z}_n$ and $S \neq \frac{1}{n}\mathbb{Z}/\mathbb{Z}$.

(ii) (Myhill property) Each non-zero generic interval comes in precisely two specific sizes.

(iii) The ratio $\dfrac{m}{n}$ is a semiconvergent of the generator g with $\dfrac{m}{n} \neq g$.

Example $g = \mathrm{Log}_2(3/2)$:
with semiconvergents 1/2, 2/3, 3/5, 4/7, 7/12, ...

Is the automorphism musically relevant?

$$\cdot m : \mathbb{Z}_n \to \mathbb{Z}_n$$

$$\begin{array}{ccc} & \mathbb{Z}_n & \\ \cdot m & \downarrow\wr & \searrow\!\!\sim \\ & \mathbb{Z}_n \xrightarrow{\sim} S \subset \mathbb{R}/\mathbb{Z} & \end{array}$$

The diatonic case:

n = 7

m = 4

Every step can be divided into two fifths.
Every fifth can be divided into four steps.

4 . 2 = 1 mod 7

Excerpt from the madrigal "Questi vaghi"

Claudio Monteverdi

Beginning of the “Passacaille”
(Suite for Harpsichord in G-Minor)

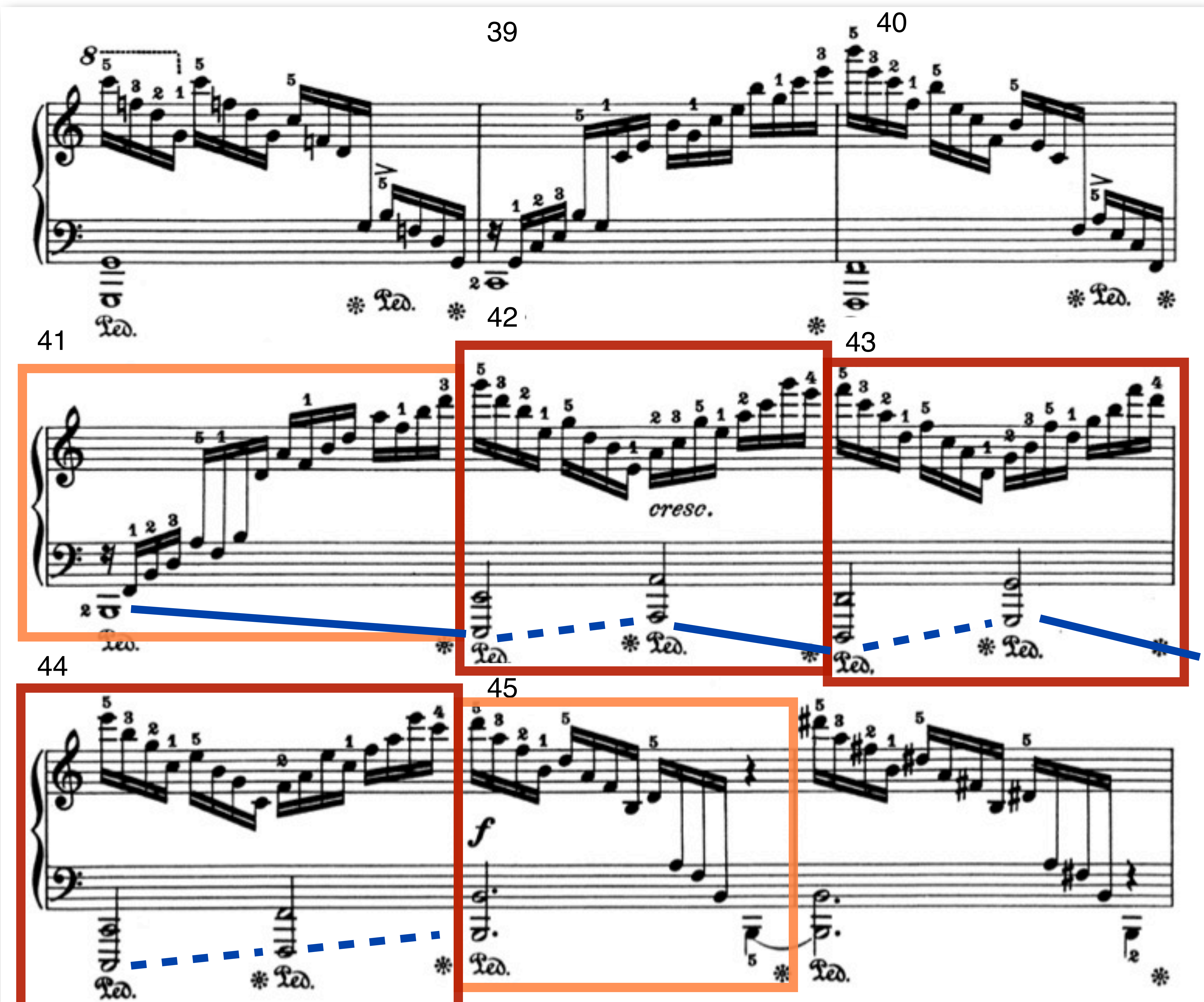
Georg Friedrich Händel

1st Theme from a Piano Sonata

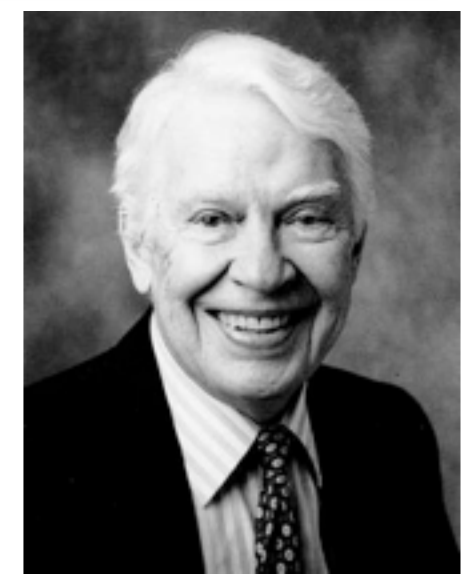
A D G C F B E A

(Fundamental Bass)

Excerpt from a Study for Piano



F. Chopin



Bart Howard

Extending the Linear Automorphism

Octave #   Fifth Fourth

$$
\begin{array}{ccccccccc}
0 & \to & \mathbb{Z}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \end{pmatrix}\right] & \hookrightarrow & \mathbb{Z}\,[P5, P4] & \xrightarrow{-} & \mathbb{Z}_7 & \to & 0 \\
 & & \downarrow \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} & & \downarrow \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} & & \downarrow \cdot 4 & & \\
0 & \to & \mathbb{Z}\left[\begin{pmatrix} 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] & \hookrightarrow & \mathbb{Z}\,[M2, m2] & \xrightarrow{+} & \mathbb{Z}_7 & \to & 0
\end{array}
$$

Octave #   Major, minor Seconds

let the words come into play !

Diatonic Modes

Ionian

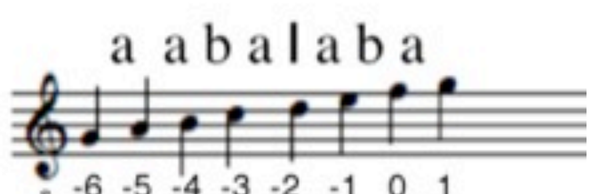
Guidonian Modes

Dorian

Phrygian

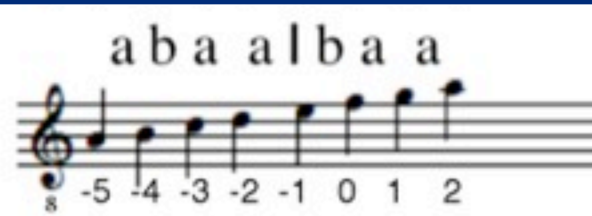
Lydian

Mixolydian

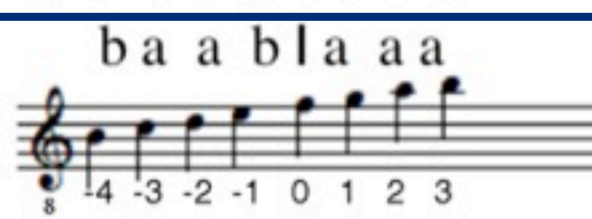


Glarean Modes

Aeolian

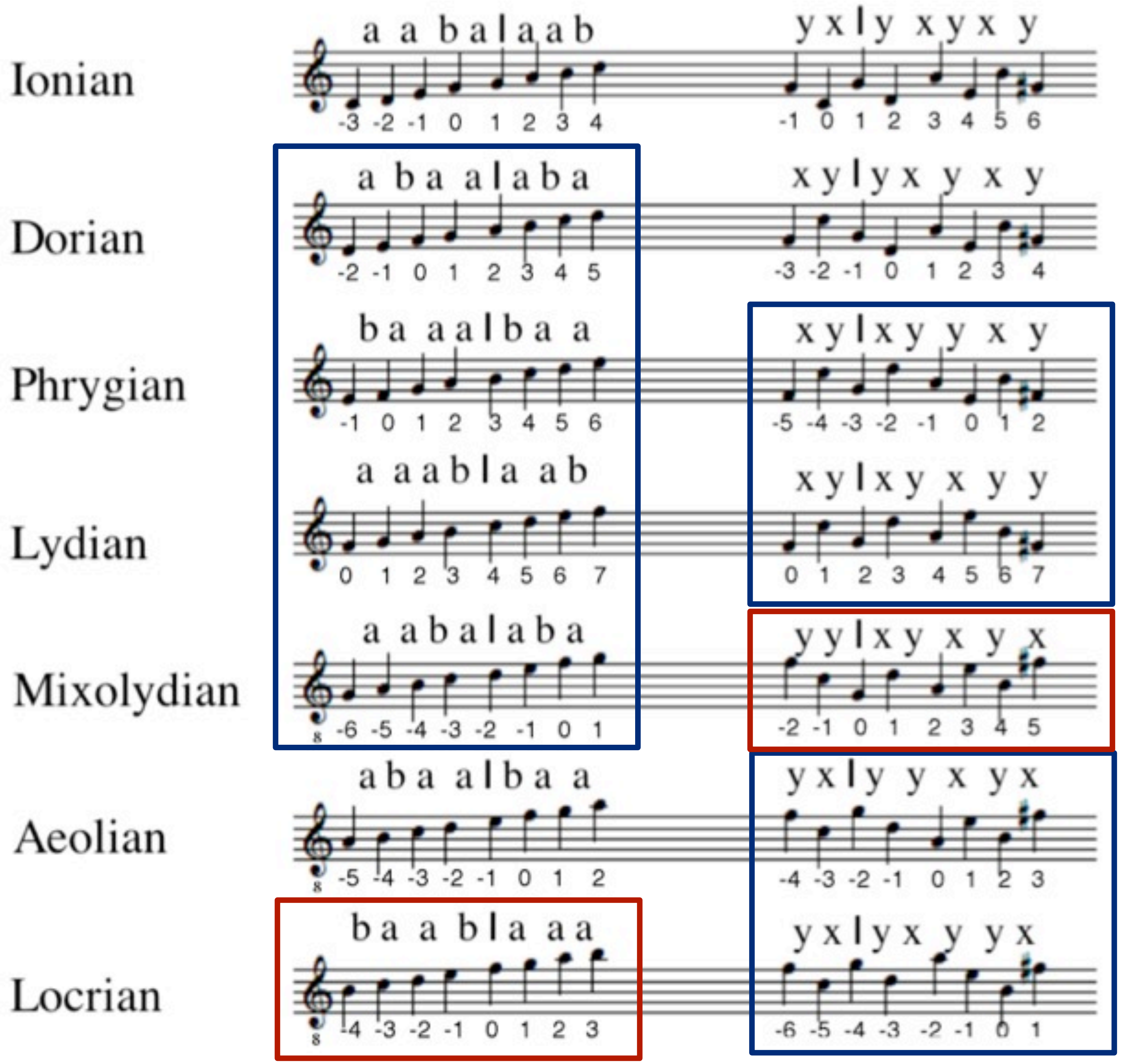


Locrian

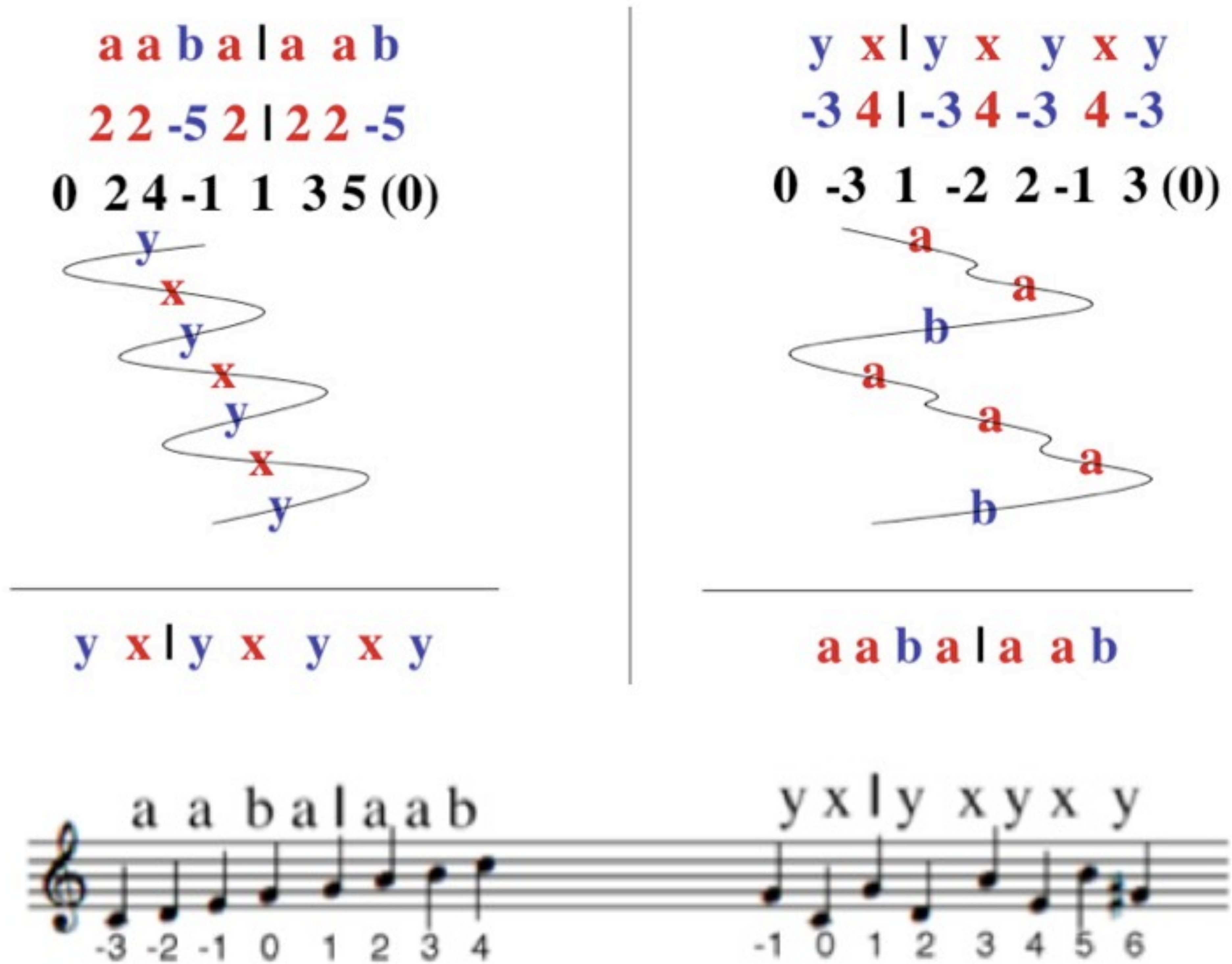


bad conjugate

Modes and their Plain Adjoints



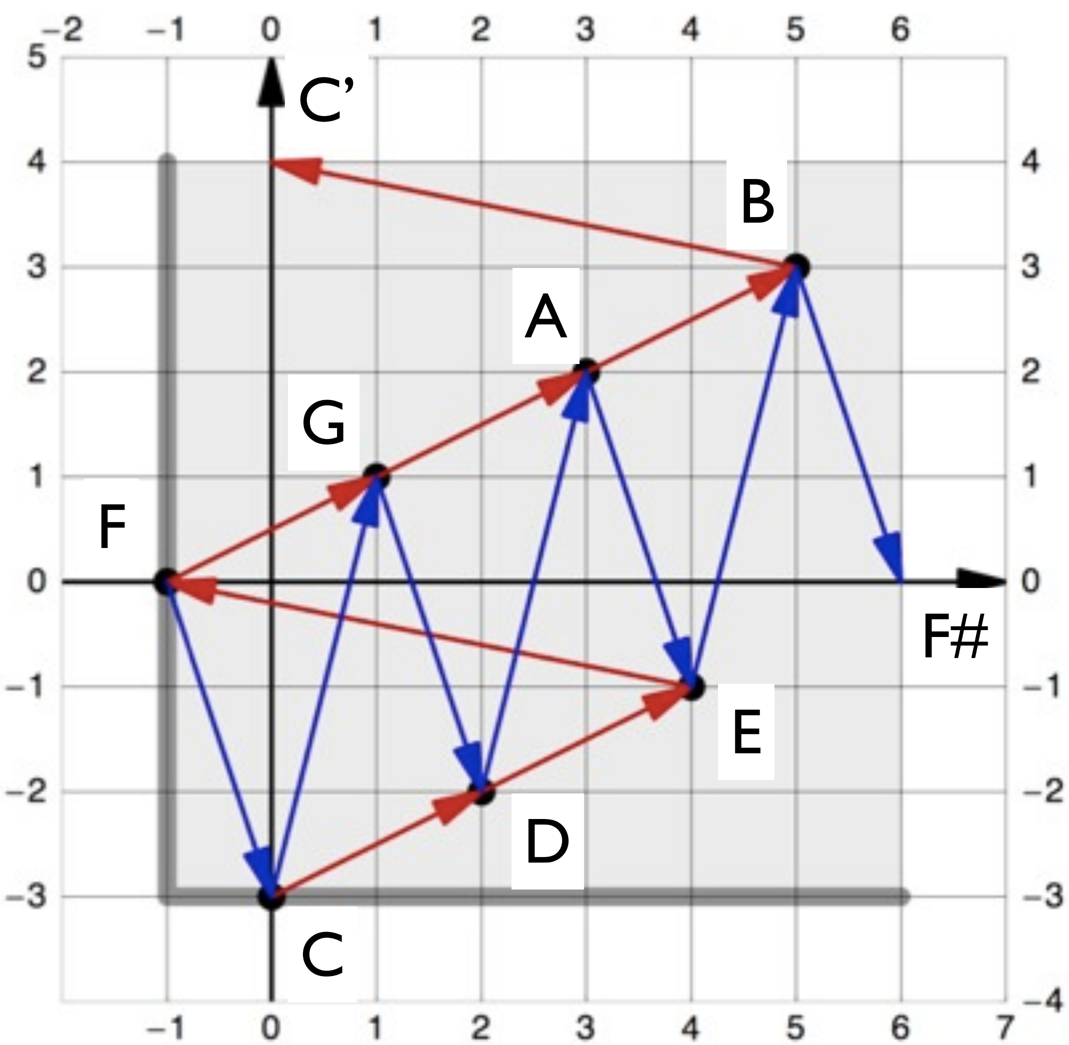
Calculation of the plain adjoint (by example)



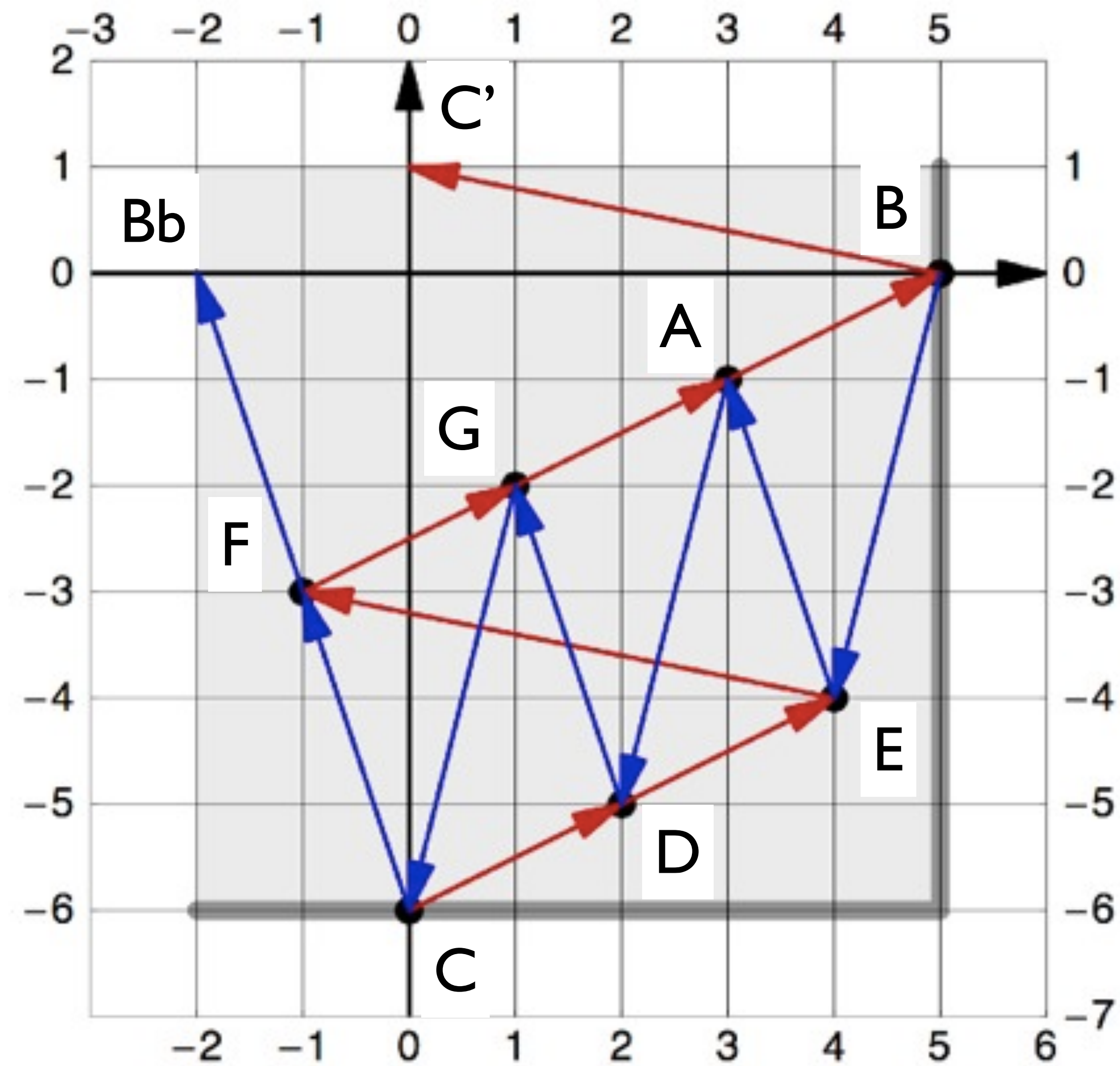
Comparing the two Adjoints in F_2

Automorphism f of F_2: $x = f(a) = aaba$ $y = f(b) = aab$

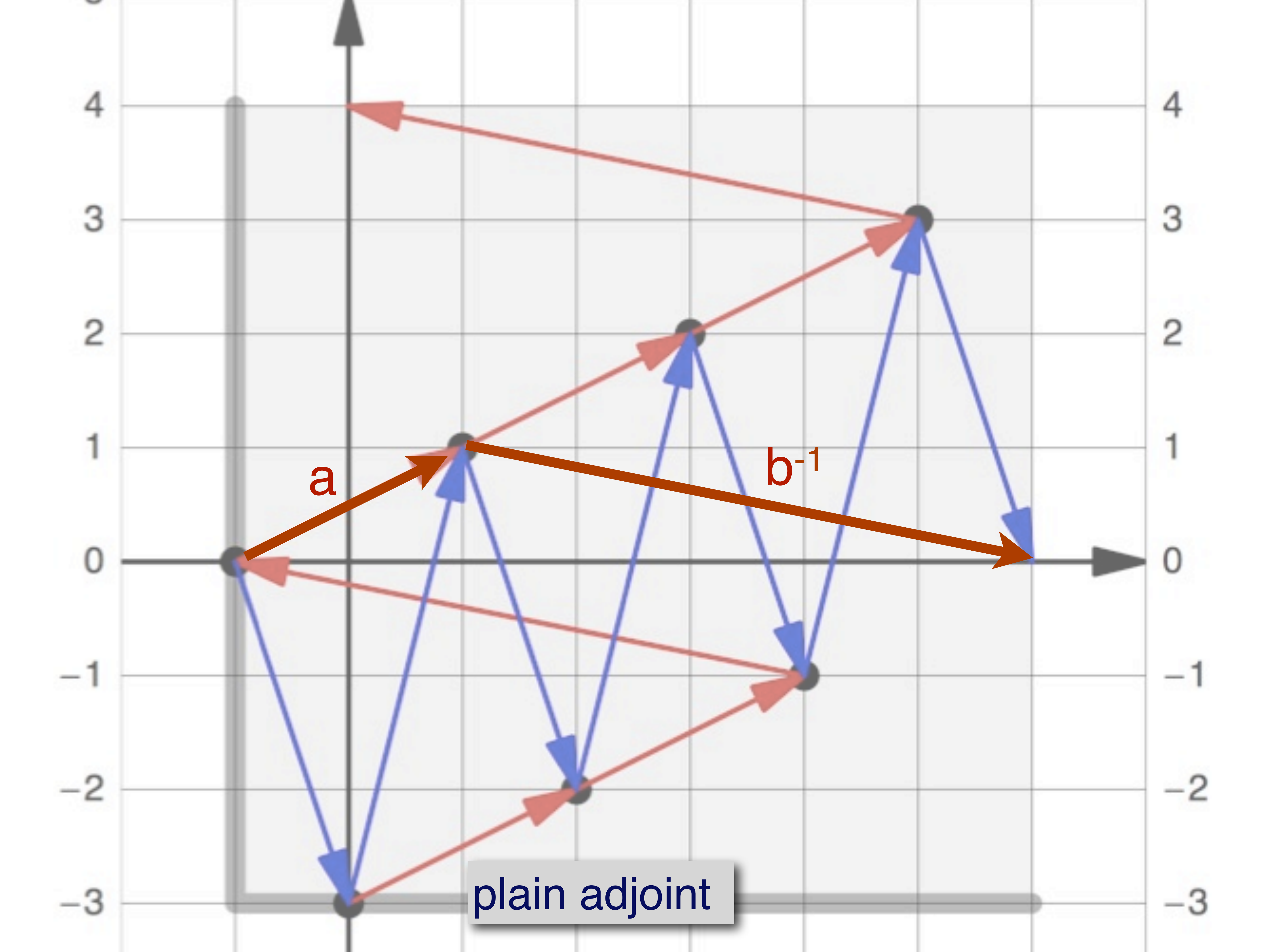
aabalaab $y^{-1}xly^{-1}xy^{-1}xy^{-1}$ aabalaab $x^{-1}ylx^{-1}yx^{-1}yy$



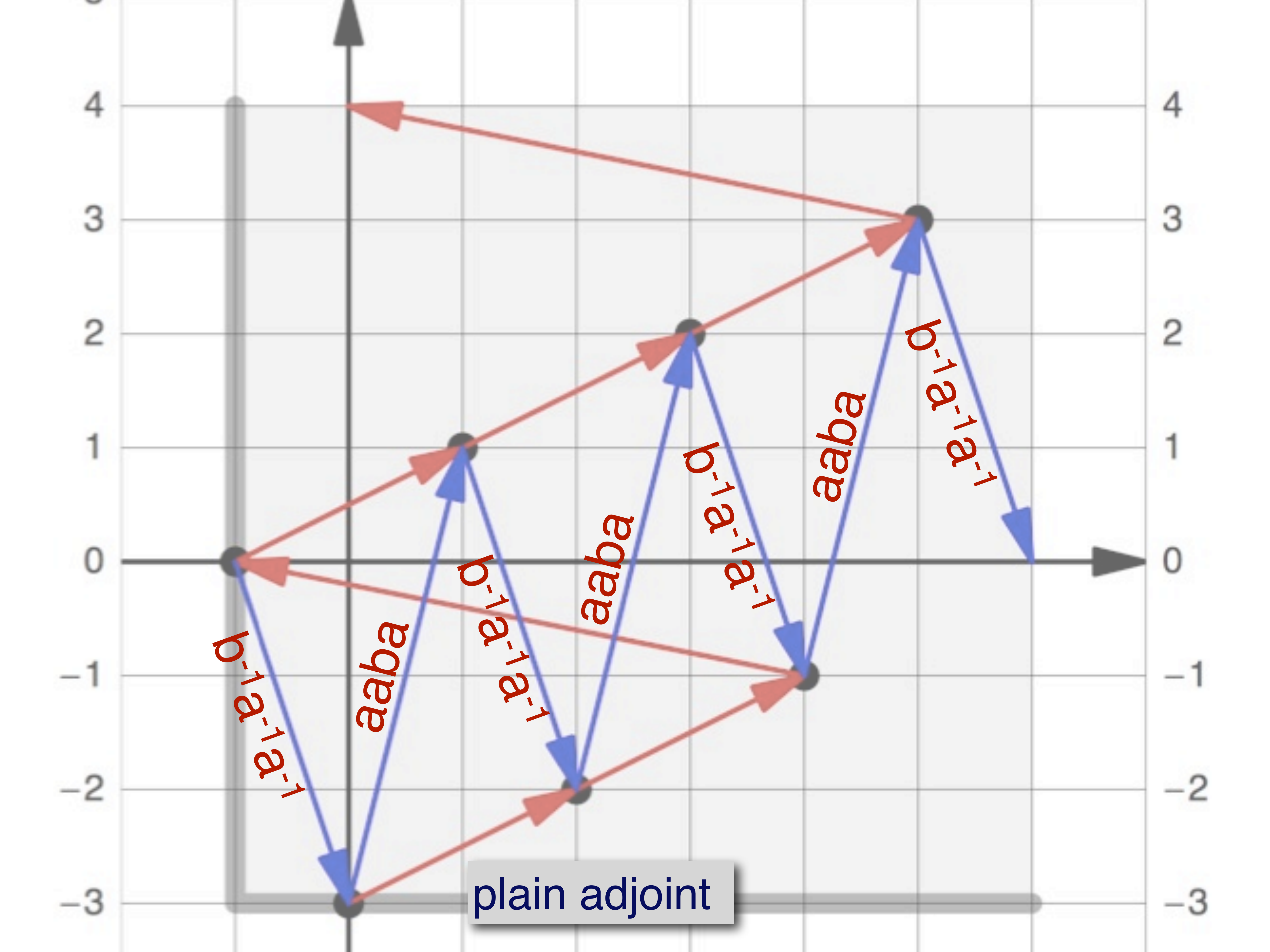
plain adjoint

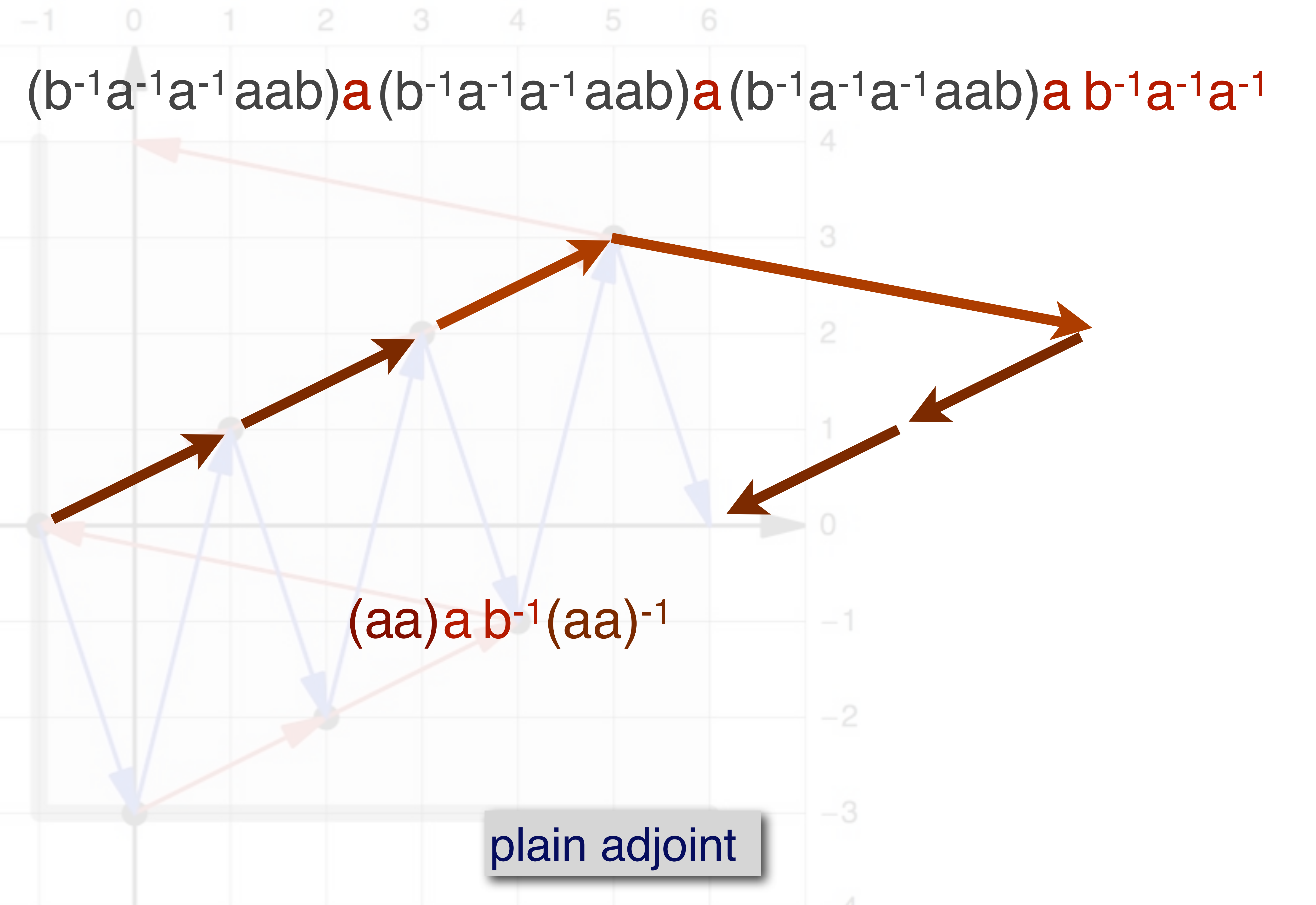


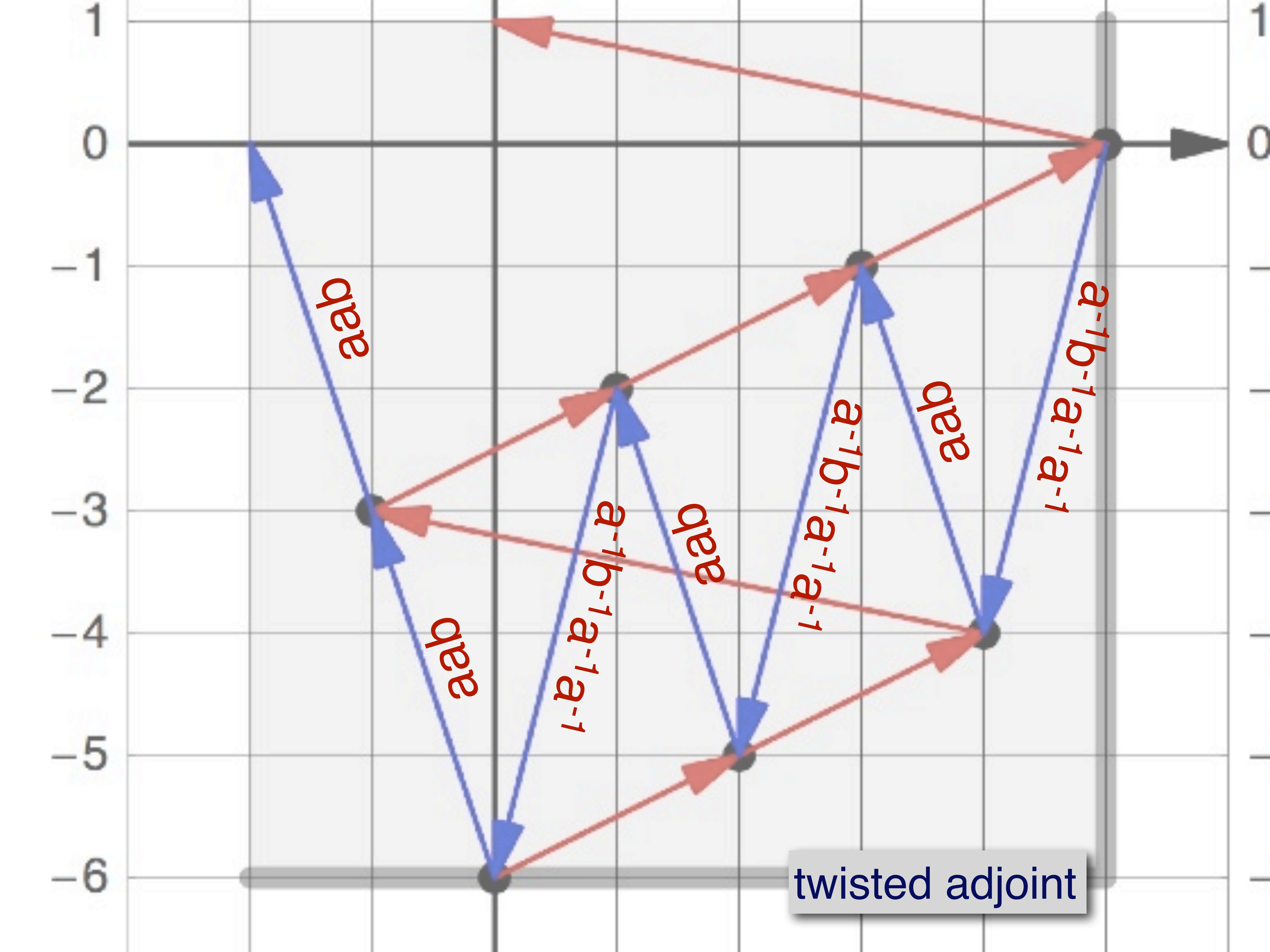
twisted adjoint



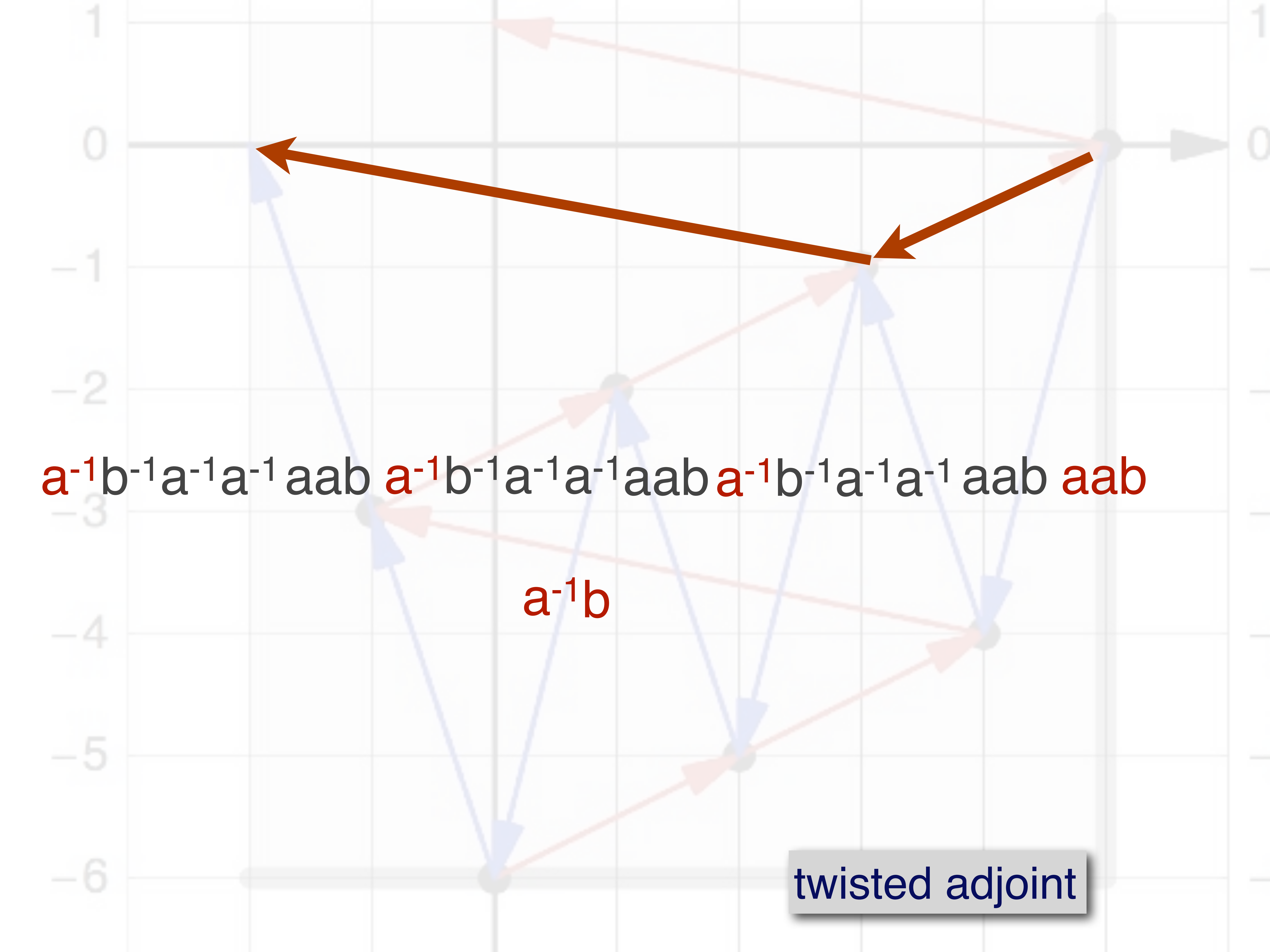
4
3
2
1
0
−1
−2
−3
a
b-1
plain adjoint


plain adjoint
aaba
b-1a-1a-1
4
3
2
1
0
−1
−2
−3


$(b^{-1}a^{-1}a^{-1}aab)a(b^{-1}a^{-1}a^{-1}aab)a(b^{-1}a^{-1}a^{-1}aab)a\,b^{-1}a^{-1}a^{-1}$
$(aa)a\,b^{-1}(aa)^{-1}$
plain adjoint
−1 0 1 2 3 4 5 6
4 3 2 1 0 −1 −2 −3



1
0
−1
−2
−3
−4
−5
−6
aab
aab
aab
aab
$a^{-1}b^{-1}a^{-1}a^{-1}$
$a^{-1}b^{-1}a^{-1}a^{-1}$
$a^{-1}b^{-1}a^{-1}a^{-1}$
twisted adjoint


1
0
−1
−2
−3
−4
−5
−6
$a^{-1}b^{-1}a^{-1}a^{-1}aab\ a^{-1}b^{-1}a^{-1}a^{-1}aab\ a^{-1}b^{-1}a^{-1}a^{-1}aab\ aab$
$a^{-1}b$
twisted adjoint

Reminder:

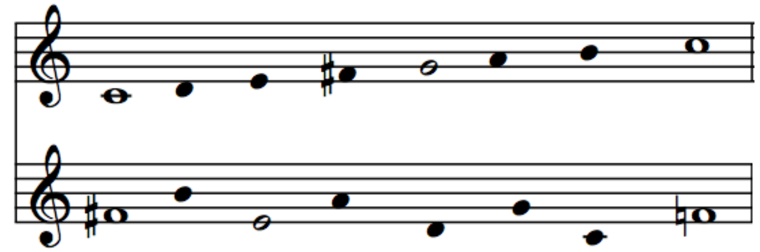
The group $Aut(F_2)$ is (redundantly) generated by the following automorphisms: $G, \tilde{G}, D, \tilde{D}, E : F_2 \to F_2$ with

$$\begin{array}{llll} G(a) = a, & G(b) = ab, & \tilde{G}(a) = \widetilde{G(a)} = a, & \tilde{G}(b) = \widetilde{G(b)} = ba, \\ D(a) = ba & D(b) = b, & \tilde{D}(a) = \widetilde{D(a)} = ab, & \tilde{D}(b) = \widetilde{D(b)} = b, \\ E(a) = b, & E(b) = a & & \end{array}$$

Proposition 4.1 *The mapping $f \mapsto f^* = (x^{-1}, y) f^{-1} (x^{-1}, y)$ is an involutive anti-automorphism of the special Sturmian monoid, that exchanges D and $\tilde{D}$ and fixes G and $\tilde{G}$. It sends conjugacy classes of morphisms onto conjugacy classes. The involution on Christoffel words that it induces is the same as the one of Section 2.*

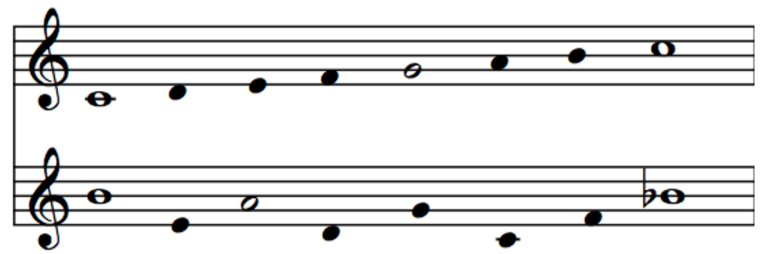
Berthé, V., de Luca, A., Reutenauer, C.: On an involution of Christoffel words and Sturmian morphisms. European Journal of Combinatorics 29(2), 535–553 (2008)

$$f_4 = GG\tilde{D} = (xxxy, xxy)$$



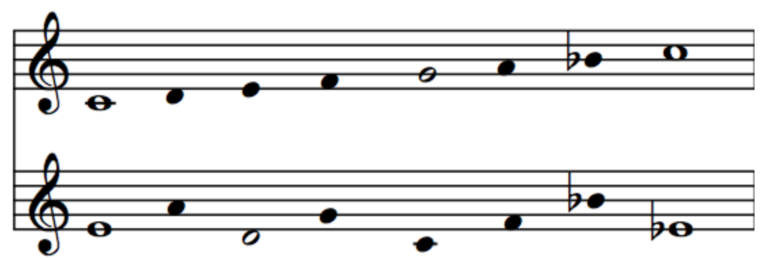
$$f_4^* = DGG = (yx, yxyxy)$$

$$f_1 = GGD = (xxyx, xxy)$$



$$f_1^* = \tilde{D}GG = (xy, xyxyy)$$

$$f_5 = G\tilde{G}\tilde{D} = (xxyx, xyx)$$



$$f_5^* = DG\tilde{G} = (yx, yxyyx)$$

$$f_2 = \tilde{G}GD = (xyxx, xyx)$$

$$f_2^* = \tilde{D}G\tilde{G} = (xy, xyyxy)$$

$$f_6 = \tilde{G}\tilde{G}\tilde{D} = (xyxx, yxx)$$

$$f_6^* = D\tilde{G}\tilde{G} = (yx, yyxyx)$$

$$f_3 = \tilde{G}\tilde{G}D = (yxxx, yxx)$$

$$f_3^* = \tilde{D}\tilde{G}\tilde{G} = (xy, yxyxy)$$

Question:

The morphisms nicely generate the interval patterns.
But we don’t have access to the notes yet.
Is there a transformational approach?

The Answer is here:

Berthé, V., de Luca, A., Reutenauer, C.: On an involution of Christoffel words and Sturmian morphisms. European Journal of Combinatorics 29(2), 535–553 (2008)

Arnoux, P., Shunji, I.: Pisot substitutions and Rauzy fractals. Bulletin of the Belgian Mathematical Society Simon Stevin 8, 181–207 (2001)

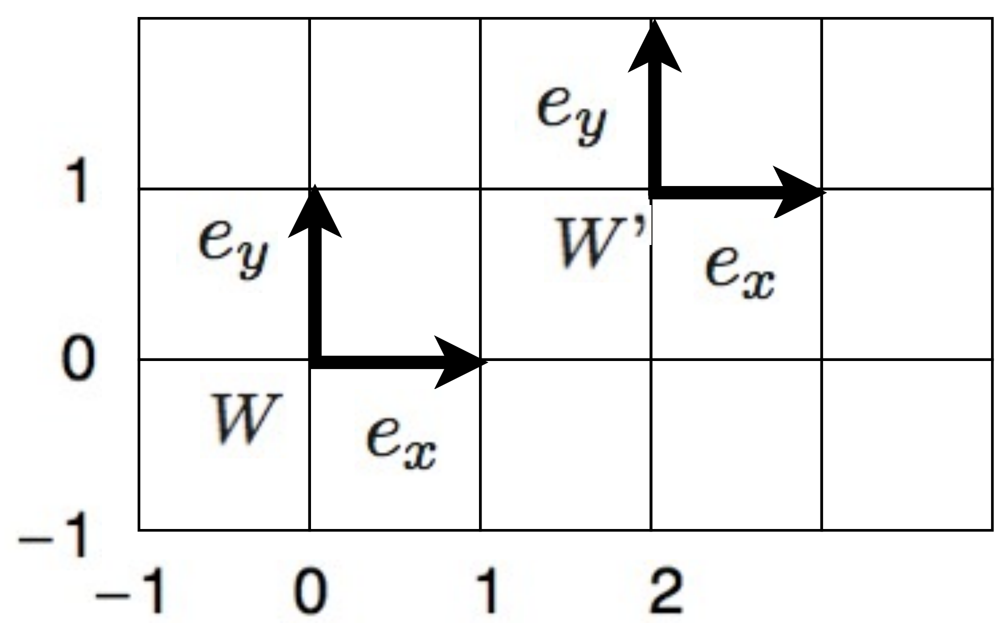




Sturmian Morphisms generate Lattice-path Transformations

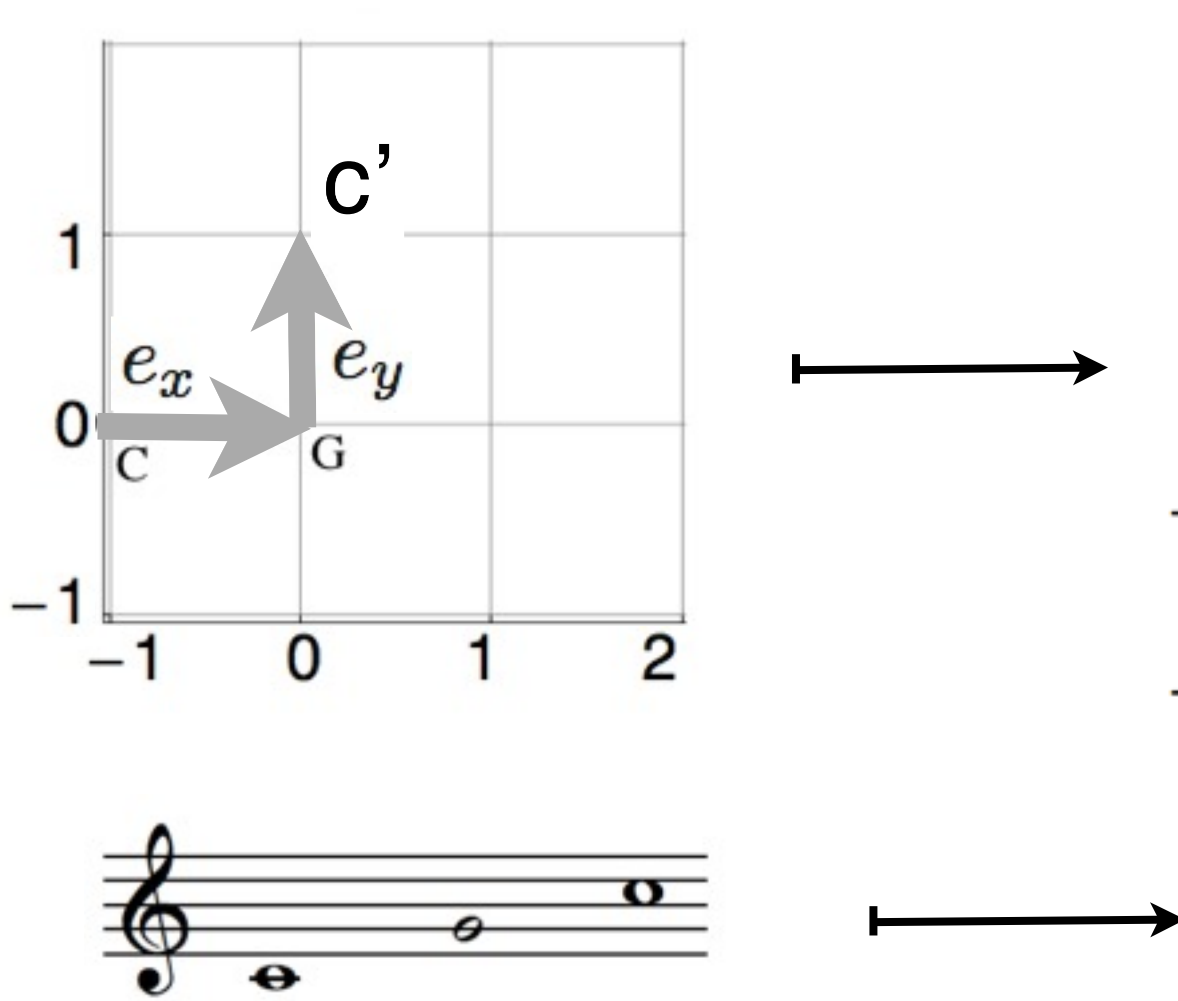
Let $e_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = V(x)$ and $e_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = V(y)$ denote the base vectors of $\mathbb{Z}^2$. Consider the set $\mathcal{B} = \{(W, e_x) \,|\, W \in \mathbb{Z}^2\} \sqcup \{(W, e_y) \,|\, W \in \mathbb{Z}^2\}$ and consider the linear space

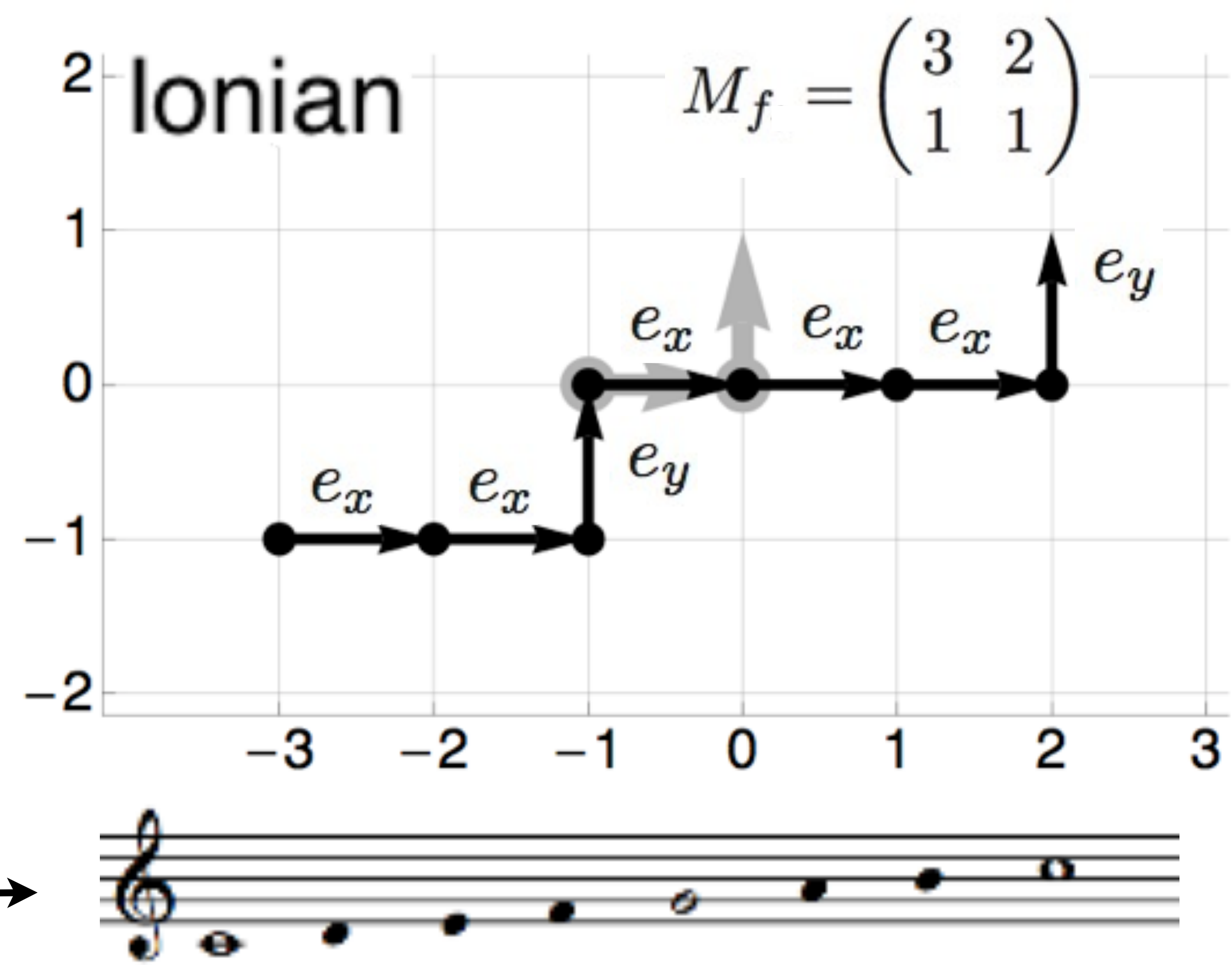
$$\mathcal{F} = \{v : \mathcal{B} \to \mathbb{R} \,|\, v(W, e_z) = 0, \text{ for all but finitely many } (W, e_z) \in \mathcal{B}\} \,.$$



Sturmian Morphisms generate Lattice-path Transformations

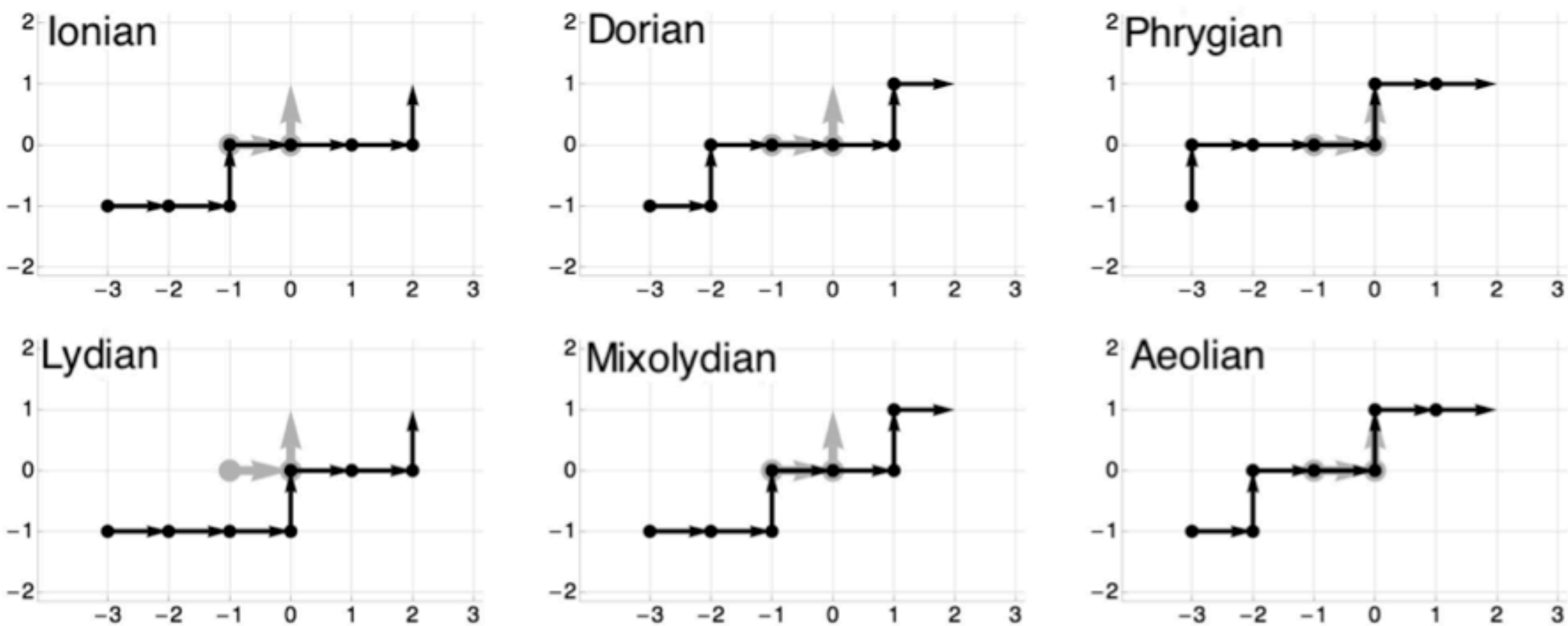
$$E(f)(W, e_z) := \sum_{k=1}^{|f(z)|} (M_f \cdot W + V(P_k(f(z))), e_{L_k(f(z))})$$





Common Finalis Modes ("Tropes"): The lattice-path transformations are applied to the same initial lattice path.

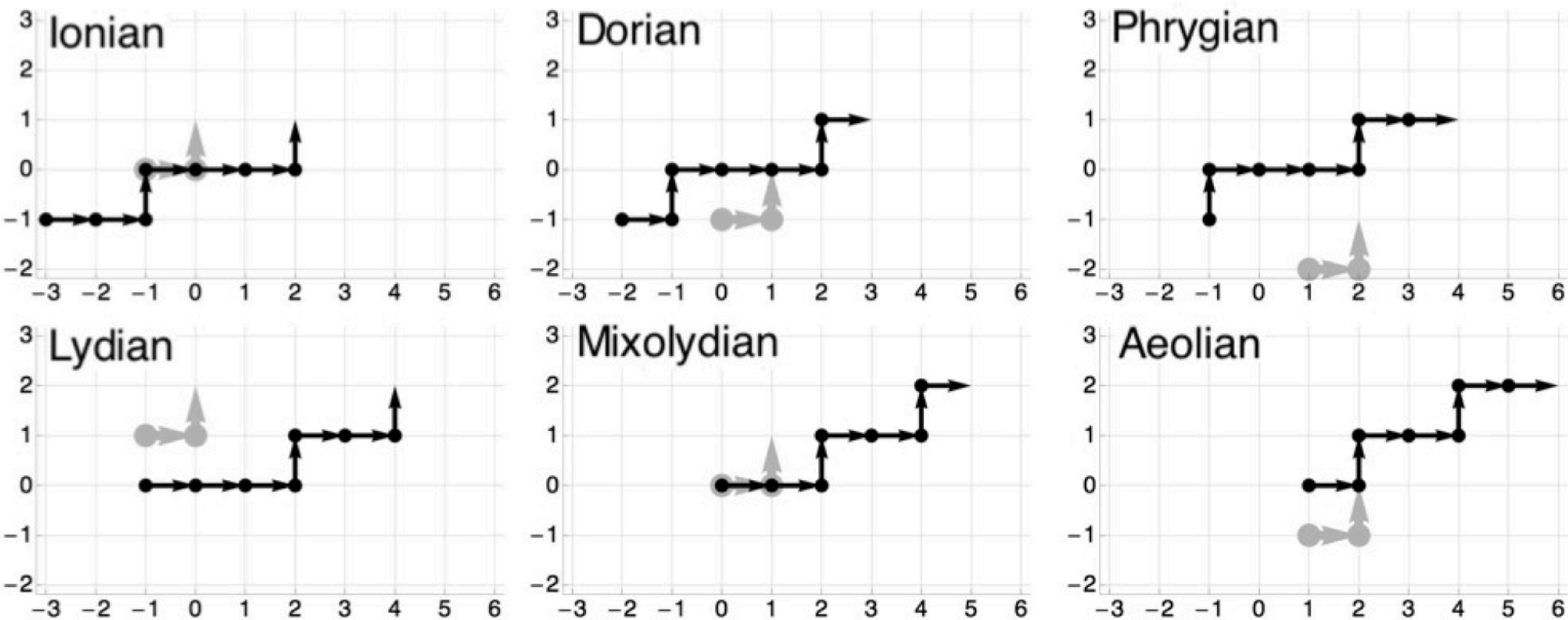
$f_1 = (xxyx, xxy)$ $f_2 = (xyxx, xyx)$ $f_3 = (yxxx, yxx)$



$f_4 = (xxxy, xxy)$ $f_5 = (xxyx, xyx)$ $f_6 = (xyxx, yxx)$

Common Origin (“White Note”) Modes:
The lattice-path transformations are applied to the different initial lattice paths.

$f_1 = (xxyx, xxy)$ $f_2 = (xyxx, xyx)$ $f_3 = (yxxx, yxx)$



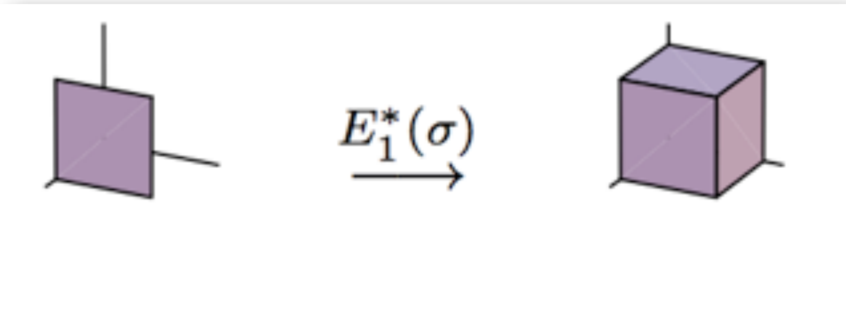
$f_4 = (xxxy, xxy)$ $f_5 = (xxyx, xyx)$ $f_6 = (xyxx, yxx)$

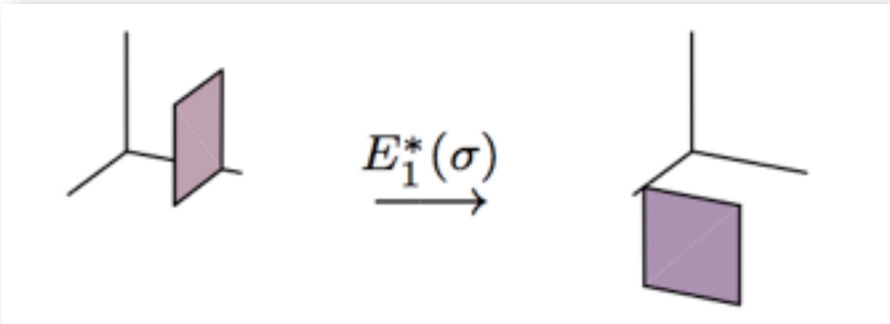
Lattice Path Transformations have Linear Adjoints

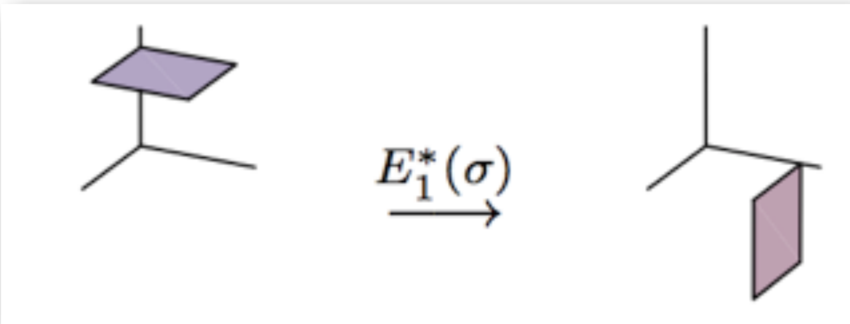
$$E(f)(W, e_z) := \sum_{k=1}^{|f(z)|} (M_f \cdot W + V(P_k(f(z))), e_{L_k(f(z))})$$

$$\begin{aligned} E(f)^*(W, e_z)^* = & \sum_{L_j(f(x))=z} (M_f^{-1}(W - V(P_j(f(x)))), e_x)^* \\ & + \sum_{L_j(f(y))=z} (M_f^{-1}(W - V(P_j(f(y)))), e_y)^* \end{aligned}$$

Geometric Interpretation (here in 3 Dimenions: Example from Arnoux and Ito)







Lattice Path Transformations have Linear Adjoints

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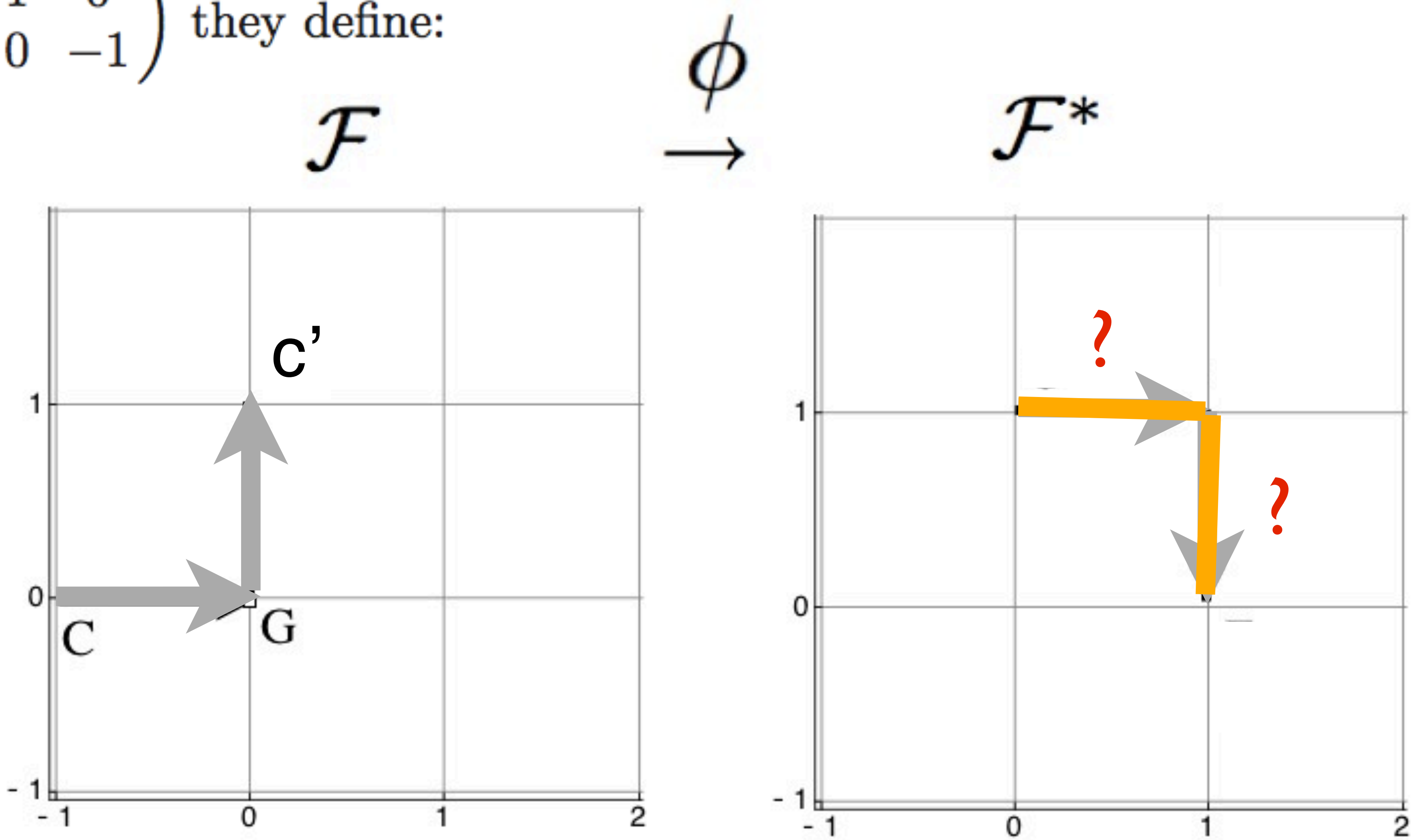
$$M_f^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

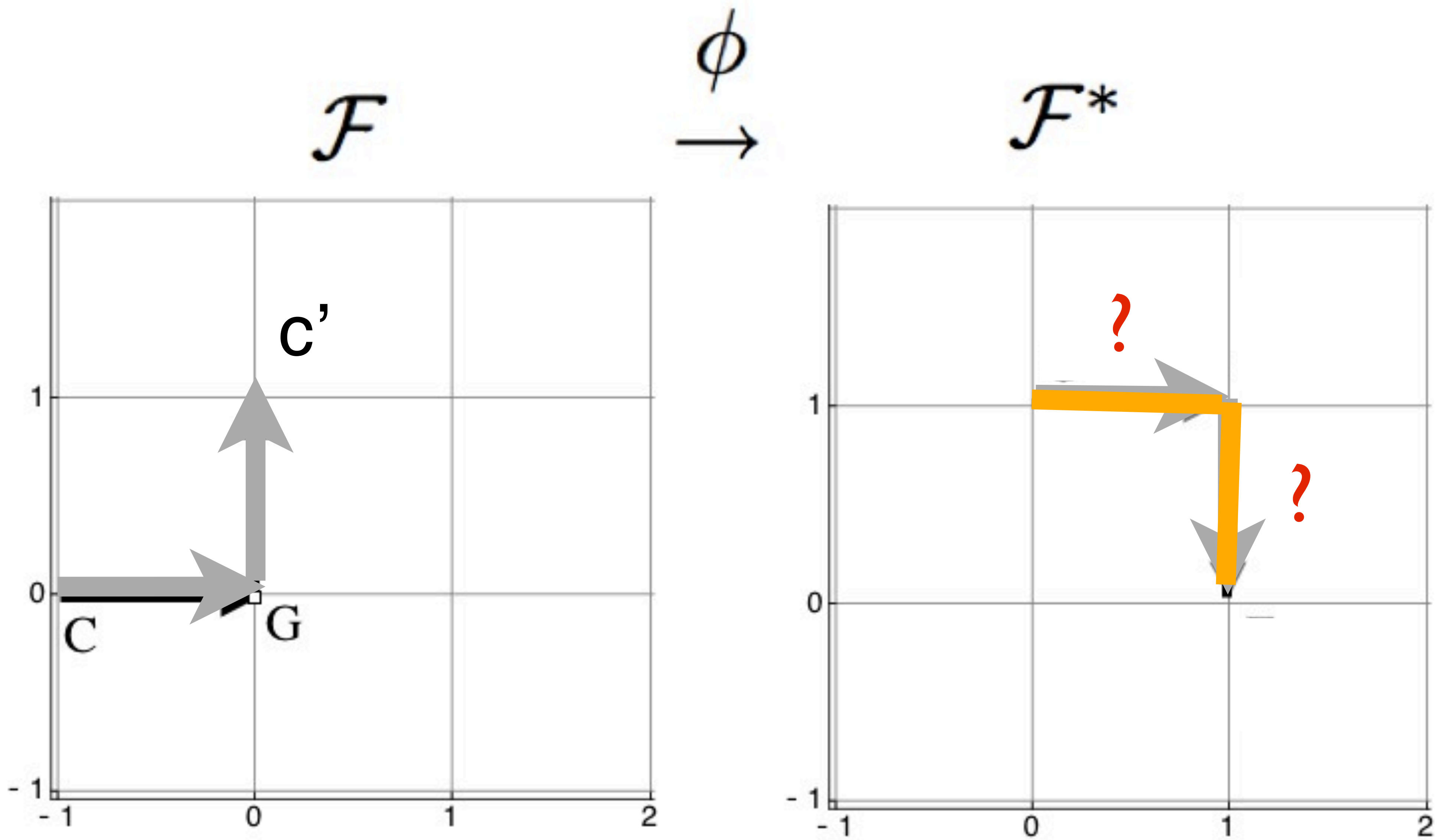
The initial lattice path can be mapped to the dual space

In order to establish a comparison between the maps $E(f)^* : \mathcal{F}^* \to \mathcal{F}^*$ and $E(f^*) : \mathcal{F} \to \mathcal{F}$ Berthe et al. [2] introduce the following map $\phi : \mathcal{F} \to \mathcal{F}^*$. With $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ they define:



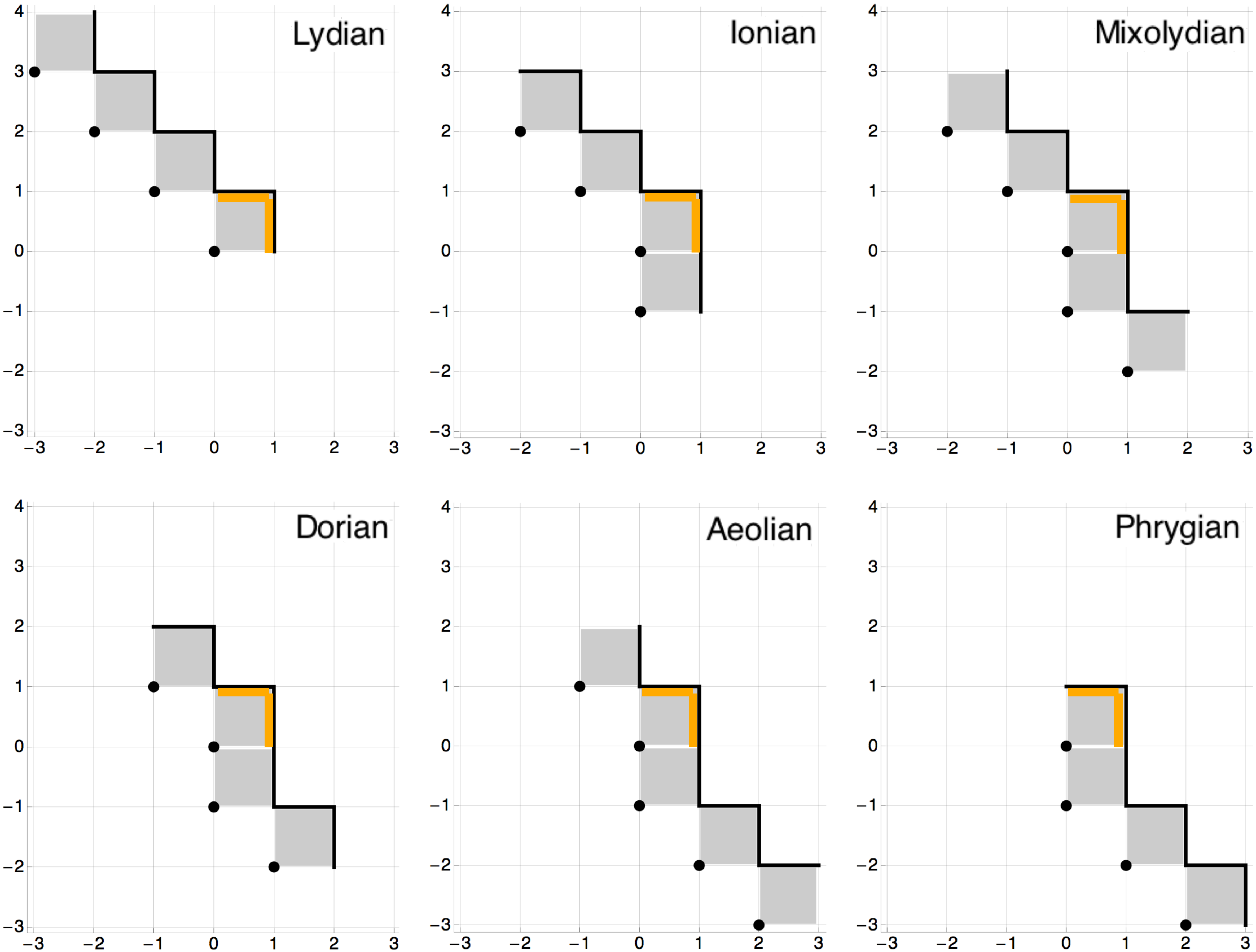
$$\phi(W, e_x) := (H \cdot W + e_x, e_y)^*, \quad \phi(W, e_y) := (H \cdot W, e_x)^* .$$

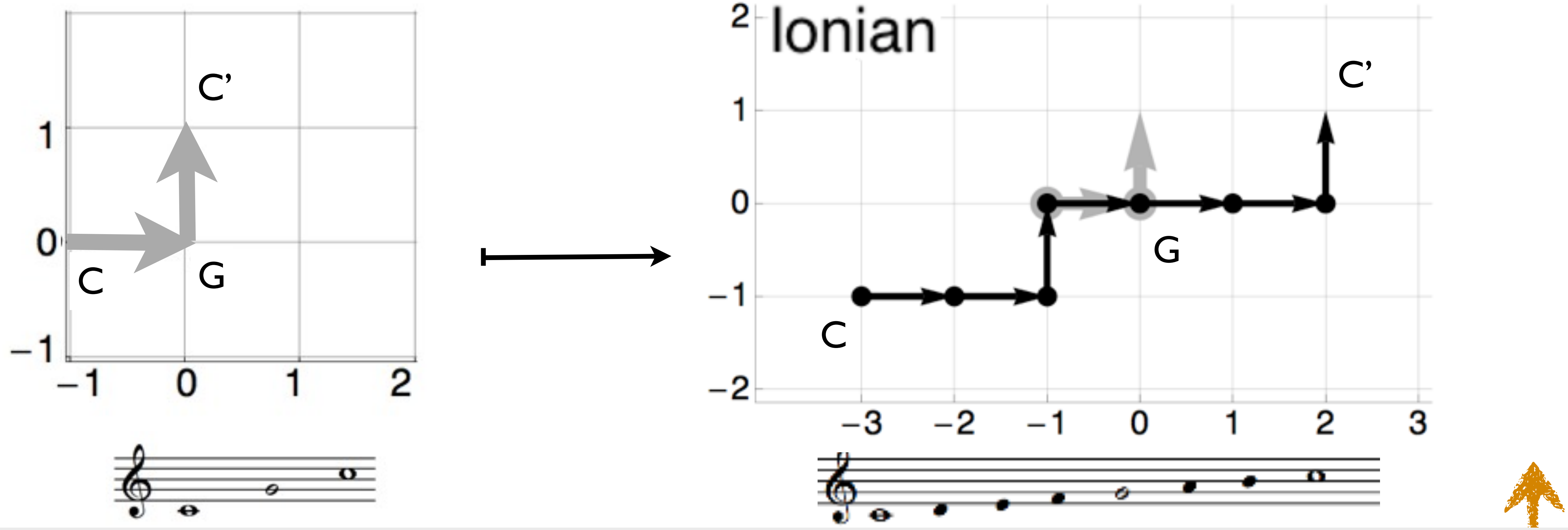
The initial lattice path can be mapped to the dual space



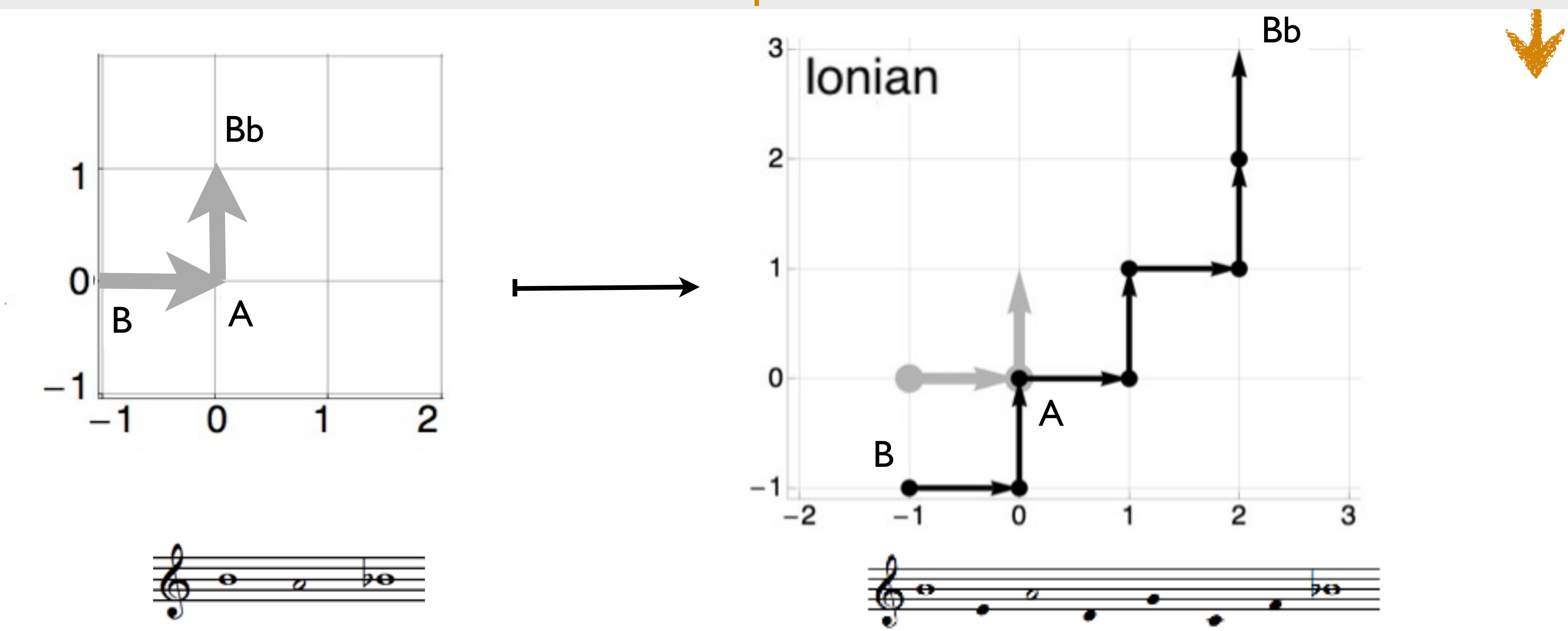
and we can apply the adjoints $E(f_i)^*$
of the 6 lattice path transformations $E(f_i)$ ...

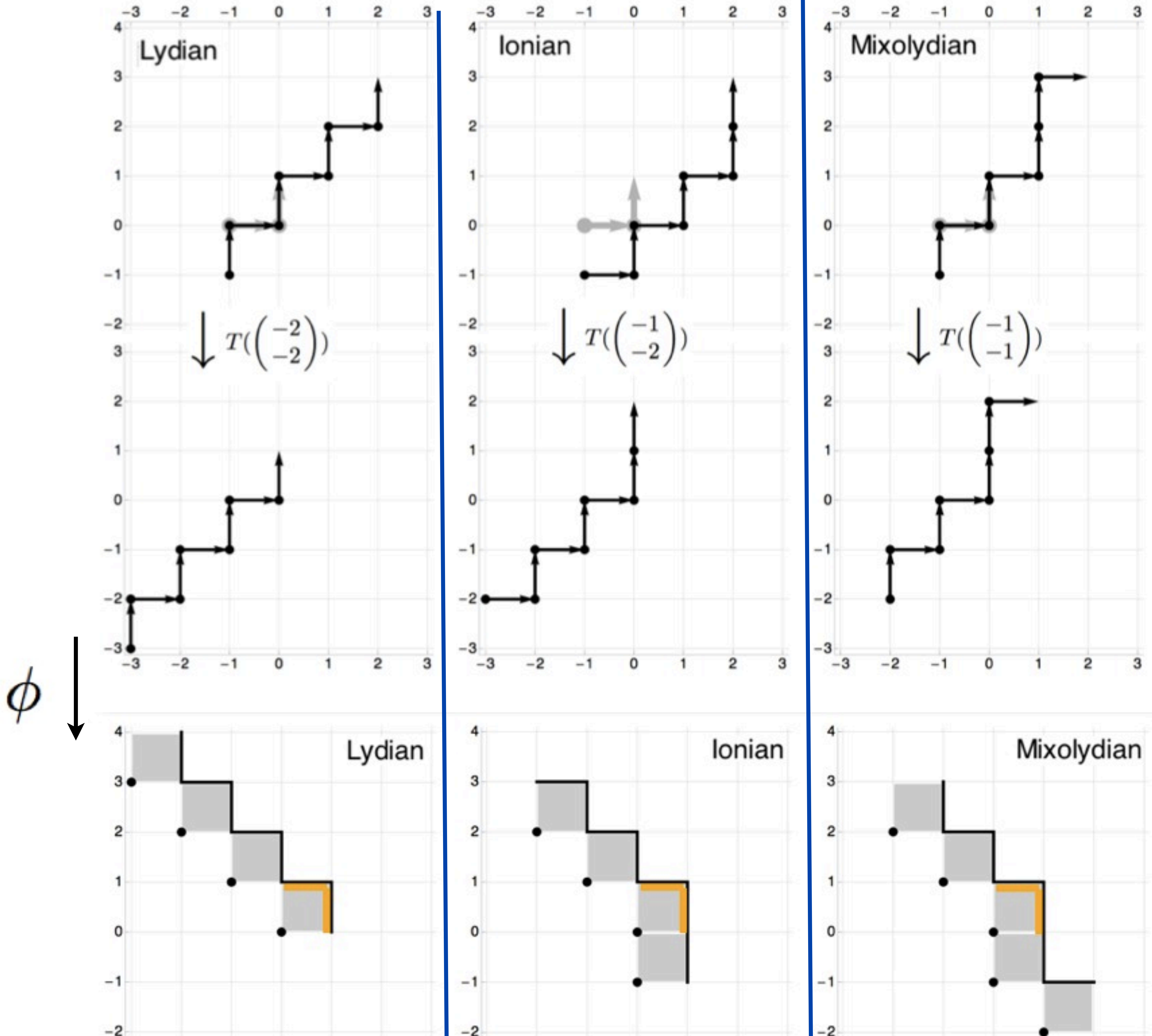
... and obtain the associated co-vectors



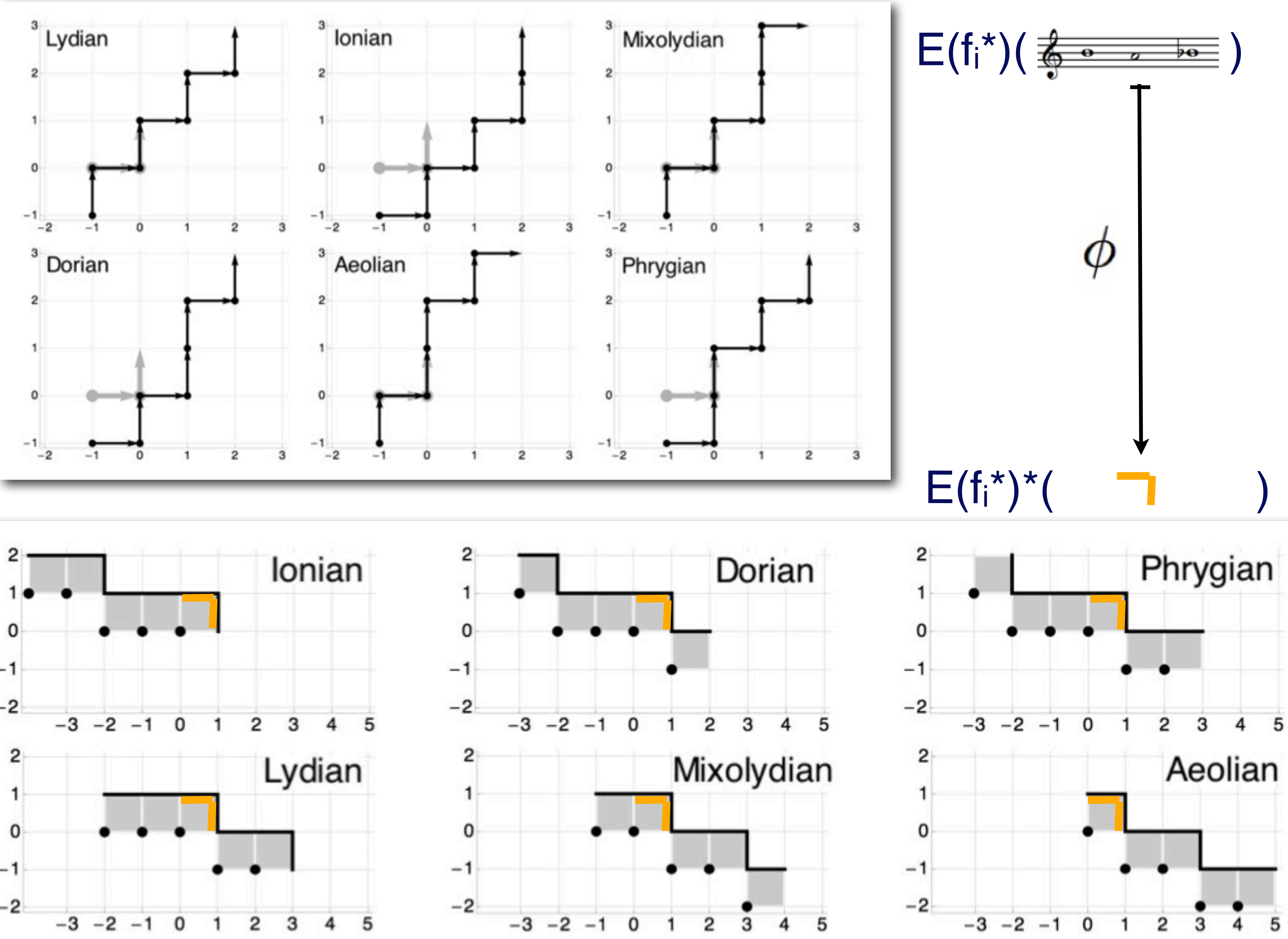


Switch music-theoretical interpretation

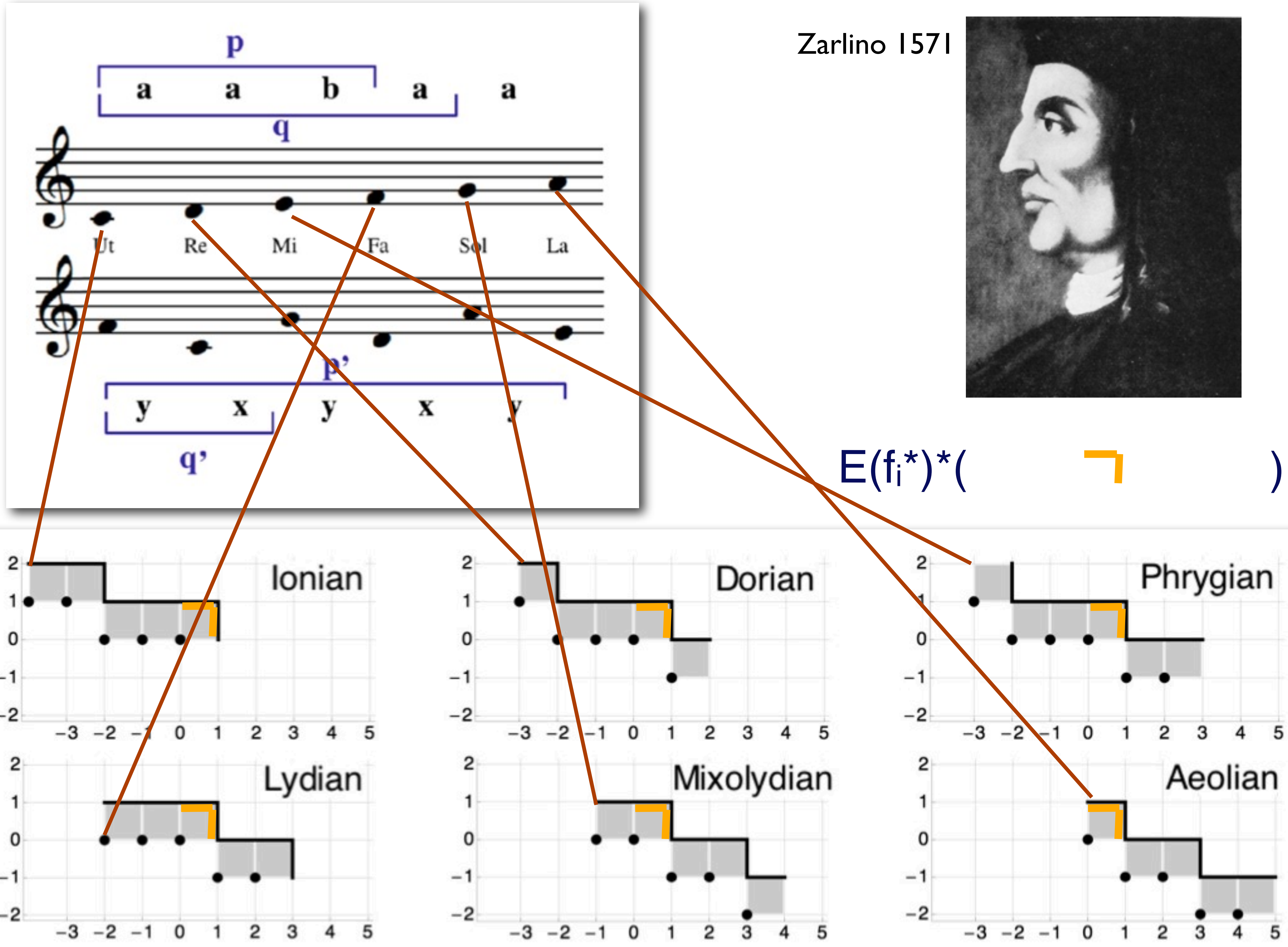




Lydian
Ionian
Mixolydian
$T(\binom{-2}{-2})$
$T(\binom{-1}{-2})$
$T(\binom{-1}{-1})$
ϕ
Lydian
Ionian
Mixolydian



Lydian
Ionian
Mixolydian
Dorian
Aeolian
Phrygian
E(fi*)( )
ϕ
E(fi*)*( )
Ionian
Dorian
Phrygian
Lydian
Mixolydian
Aeolian



Zarlino 1571
p
a a b a a
q
Ut Re Mi Fa Sol La
p'
y x y x y
q'
E(fi*)*( )
Ionian
Dorian
Phrygian
Lydian
Mixolydian
Aeolian

Ionian Substitution

$$\sigma : \{a, b\}^* \to \{a, b\}^*$$

$$a \mapsto aabb, \quad b \mapsto aab.$$

$$E_0(\sigma) = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

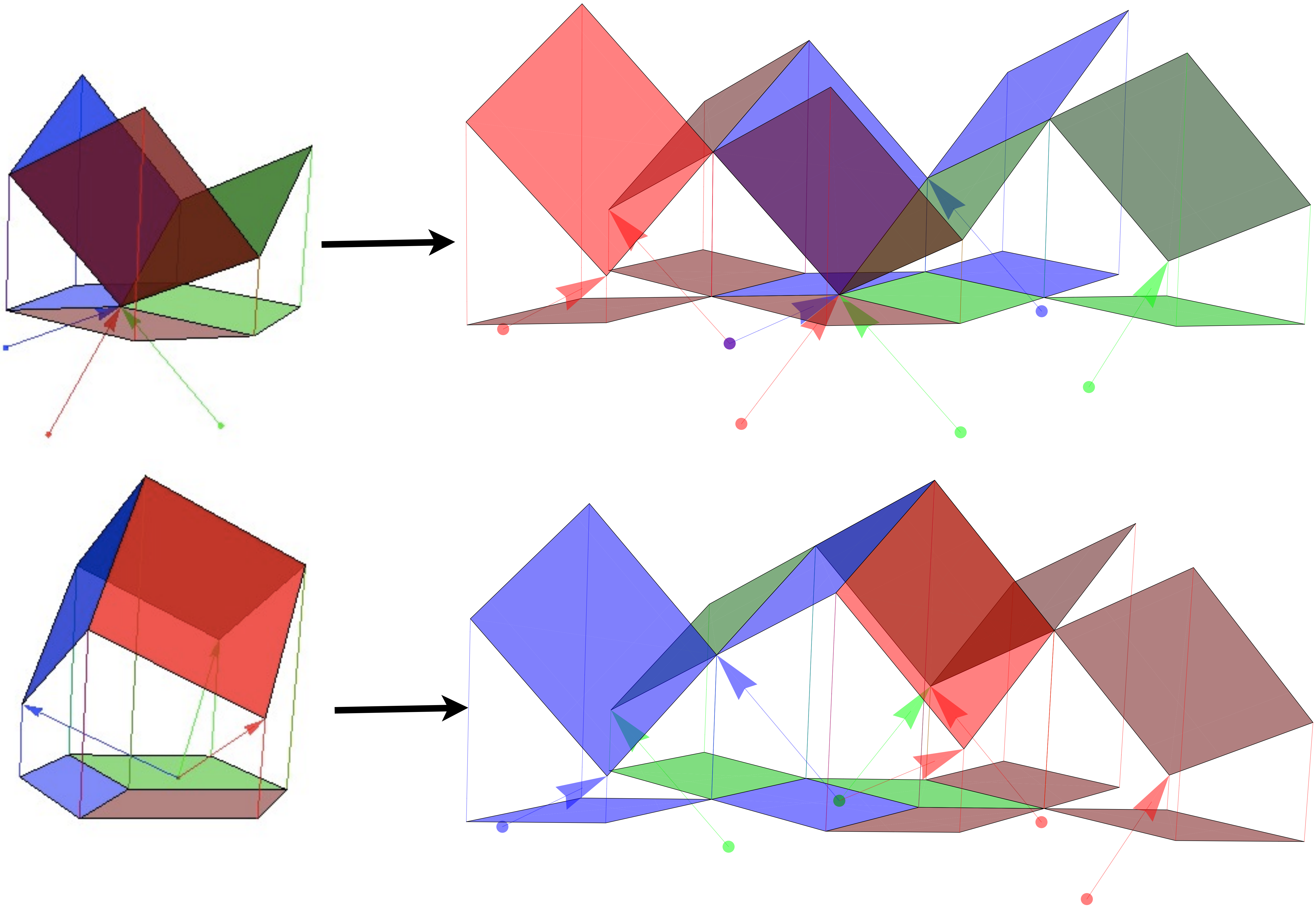
Major Substitution

$$\sigma : \{a, b, c\}^* \to \{a, b, c\}^*$$

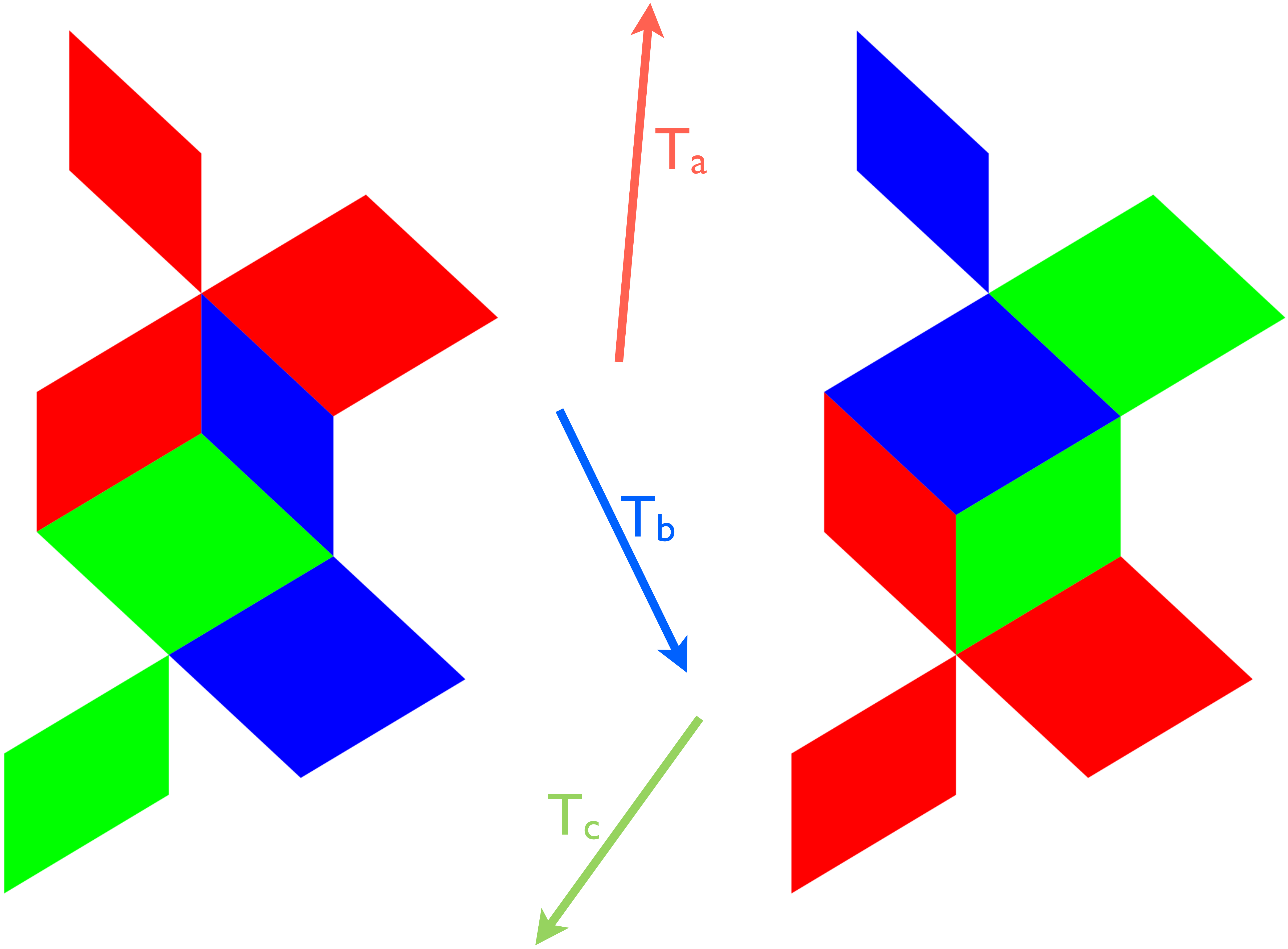
$$a \mapsto ab, \quad b \mapsto ca, \quad c \mapsto bac.$$

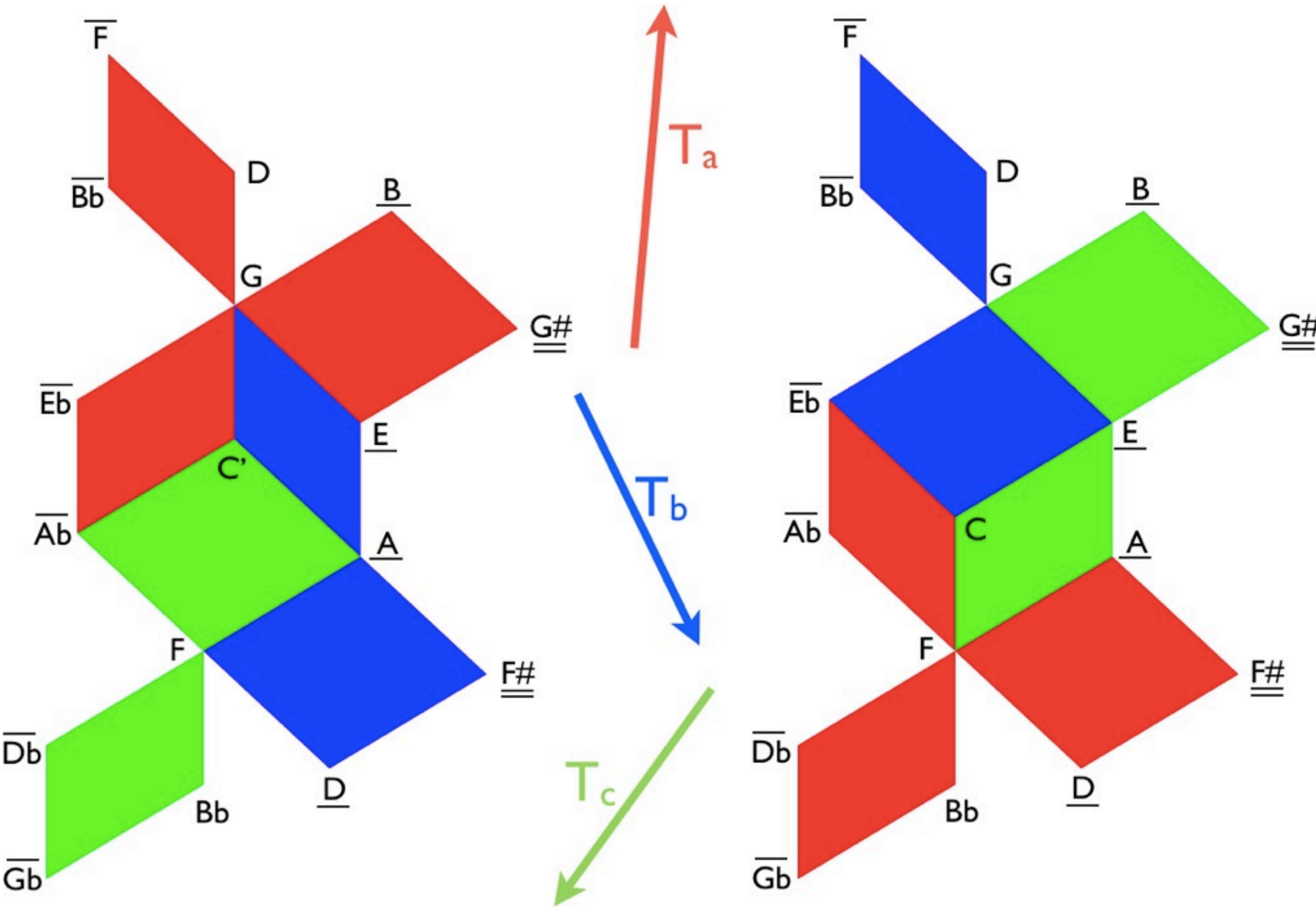
$$E_0(\sigma) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Dual Map



Dynamical System




F
D
Bb
B
G
G#
Eb
E
C'
Ab
A
F
F#
Db
Bb
D
Gb
T_a
T_b
T_c
F
D
Bb
B
G
G#
Eb
E
C
Ab
A
F
F#
Db
Bb
D
Gb