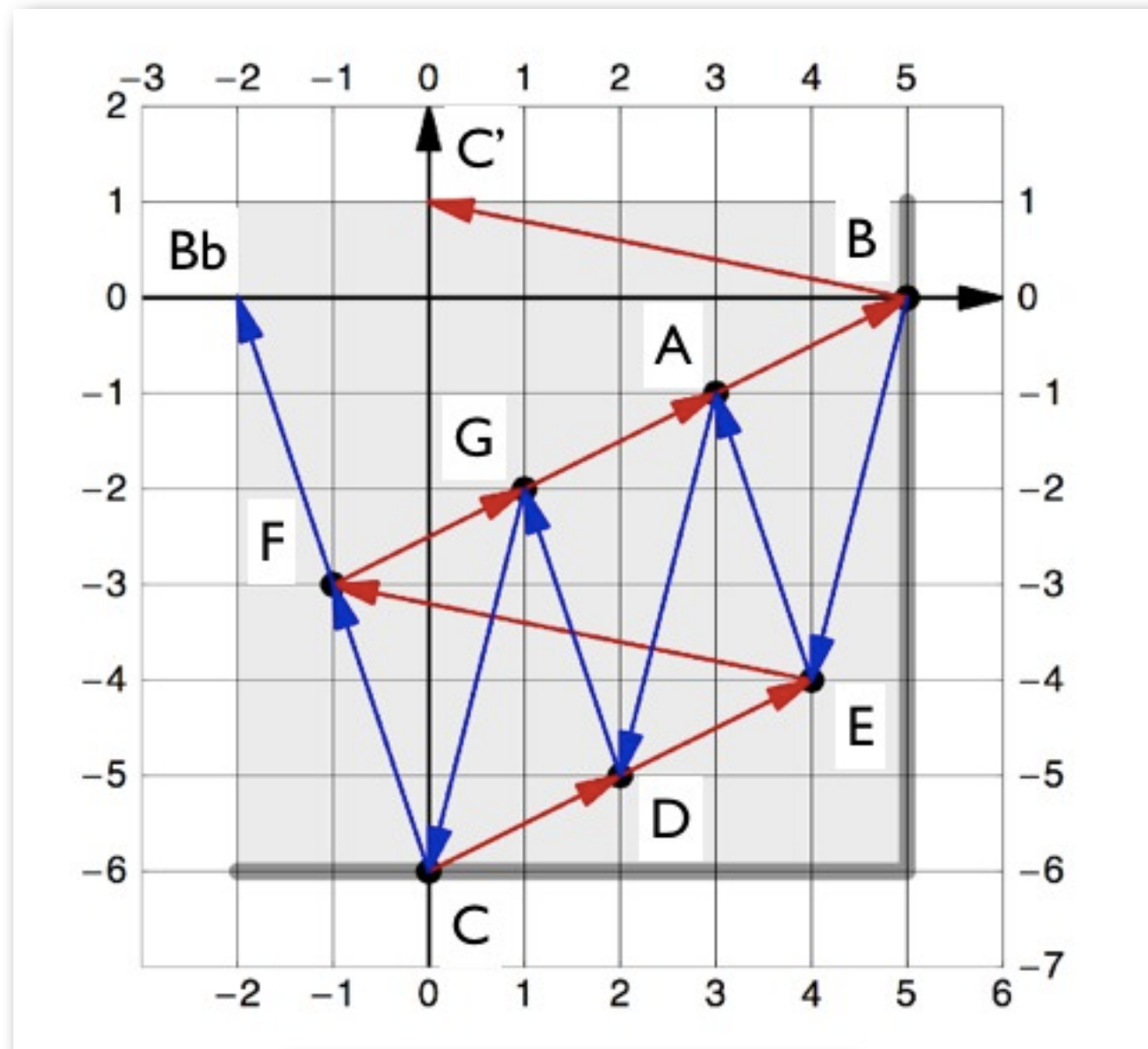
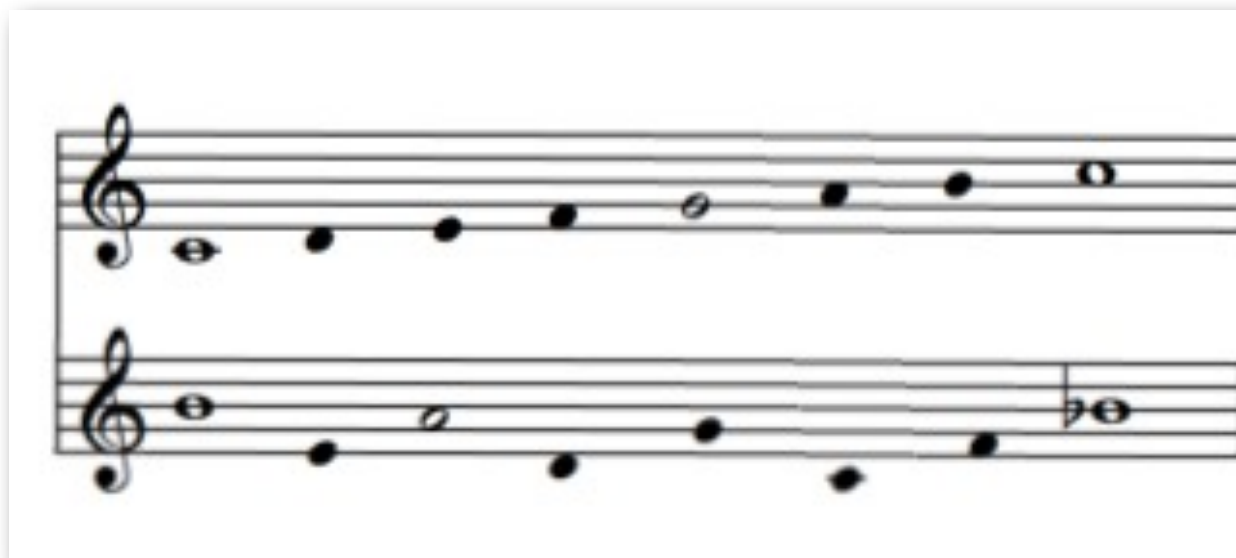


Sturmian involution, lattice-path-transformations and their application to the investigation of diatonicity

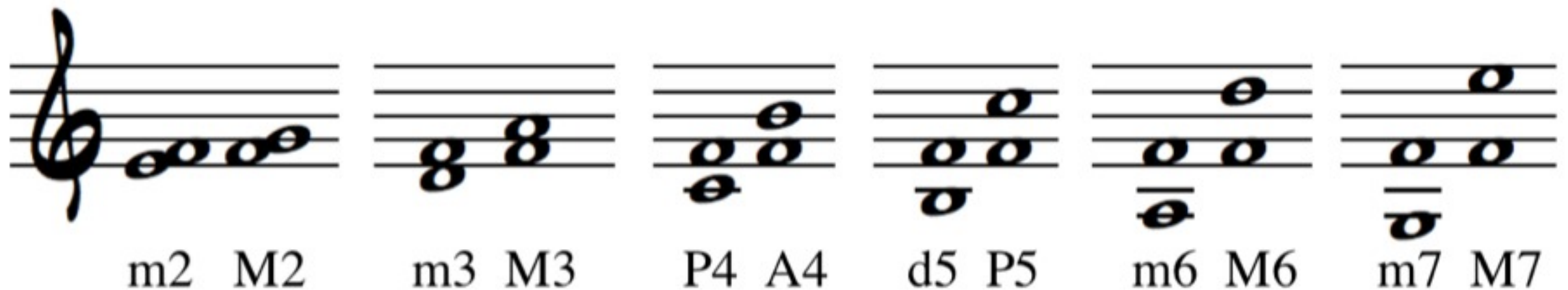
Thomas Noll

esmuc

ESCOLA SUPERIOR DE MÚSICA DE CATALUNYA



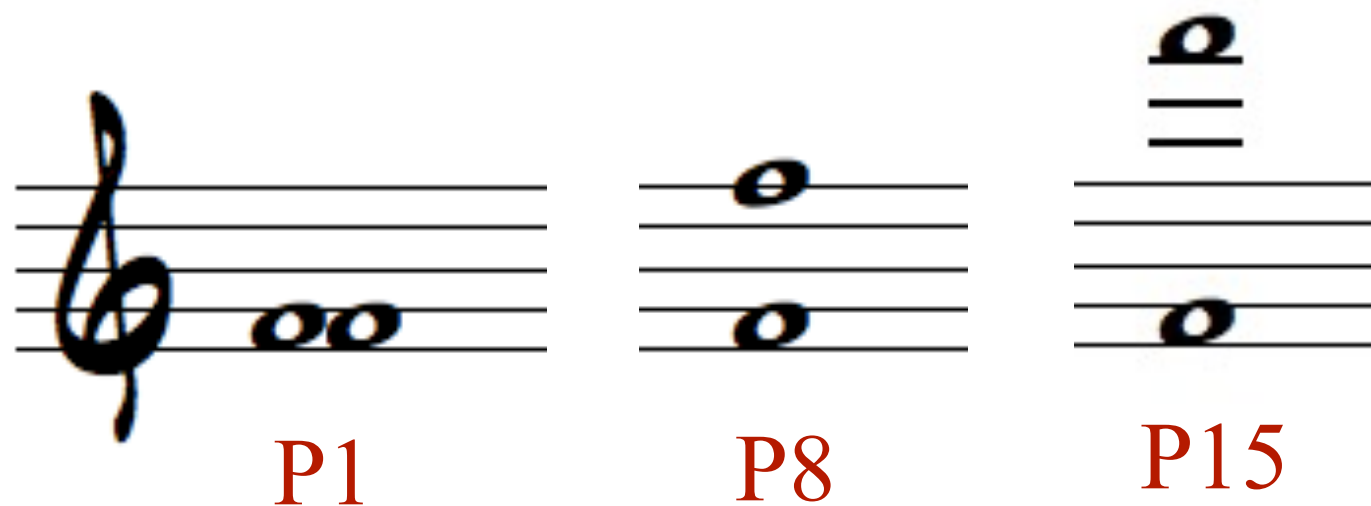
Elementary Observations about Musical Notation and the Diatonic System



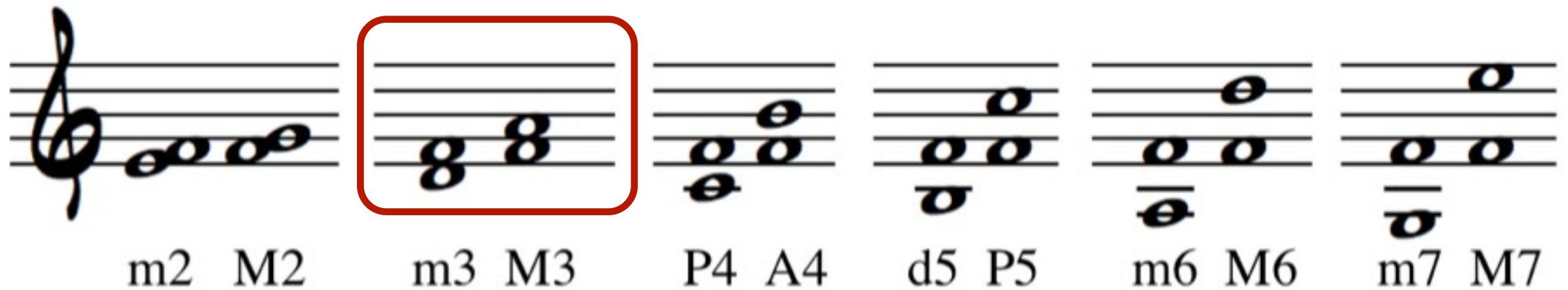
(Almost) every generic interval comes in two species.

“Myhill's Property”

Exceptions:



Elementary Observations about Musical Notation and the Diatonic System



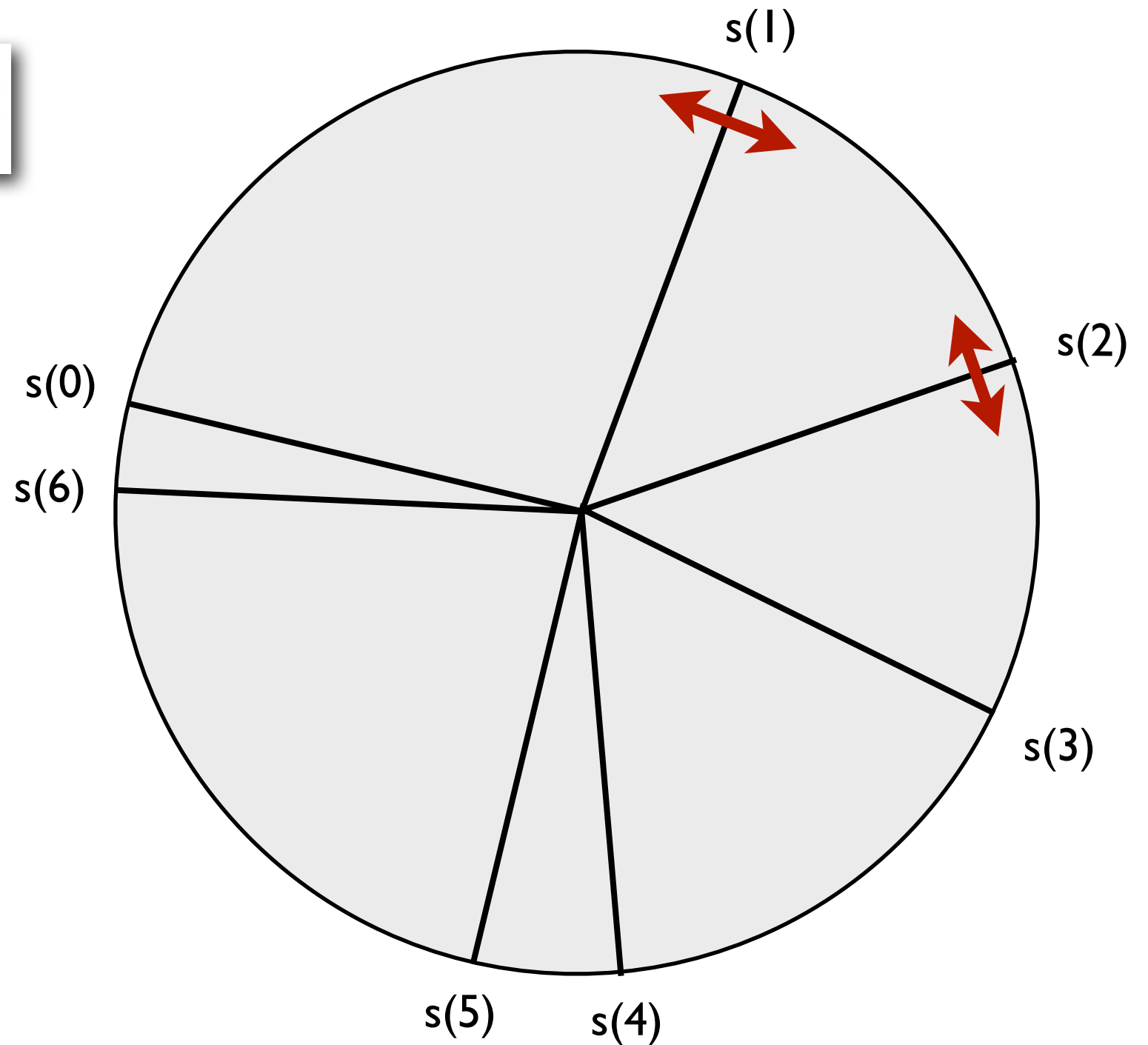
(Almost) every generic interval comes in two species.
“Myhill's Property”

The two species always differ by an alteration (# or b):



An “Arbitrary” Scale

$$s : \mathbb{Z}_n \xrightarrow{\sim} S \subset \mathbb{R}/\mathbb{Z}$$



represented in ascending ordering with:

$$s(0) < s(1) < \dots < s(n-1) < s(0) + 1$$

The Well-formedness Property

Scale:

$$s : \mathbb{Z}_n \xrightarrow{\sim} S \subset \mathbb{R}/\mathbb{Z}$$

$$s(0) < s(1) < \dots < s(n-1) < s(0) + 1$$

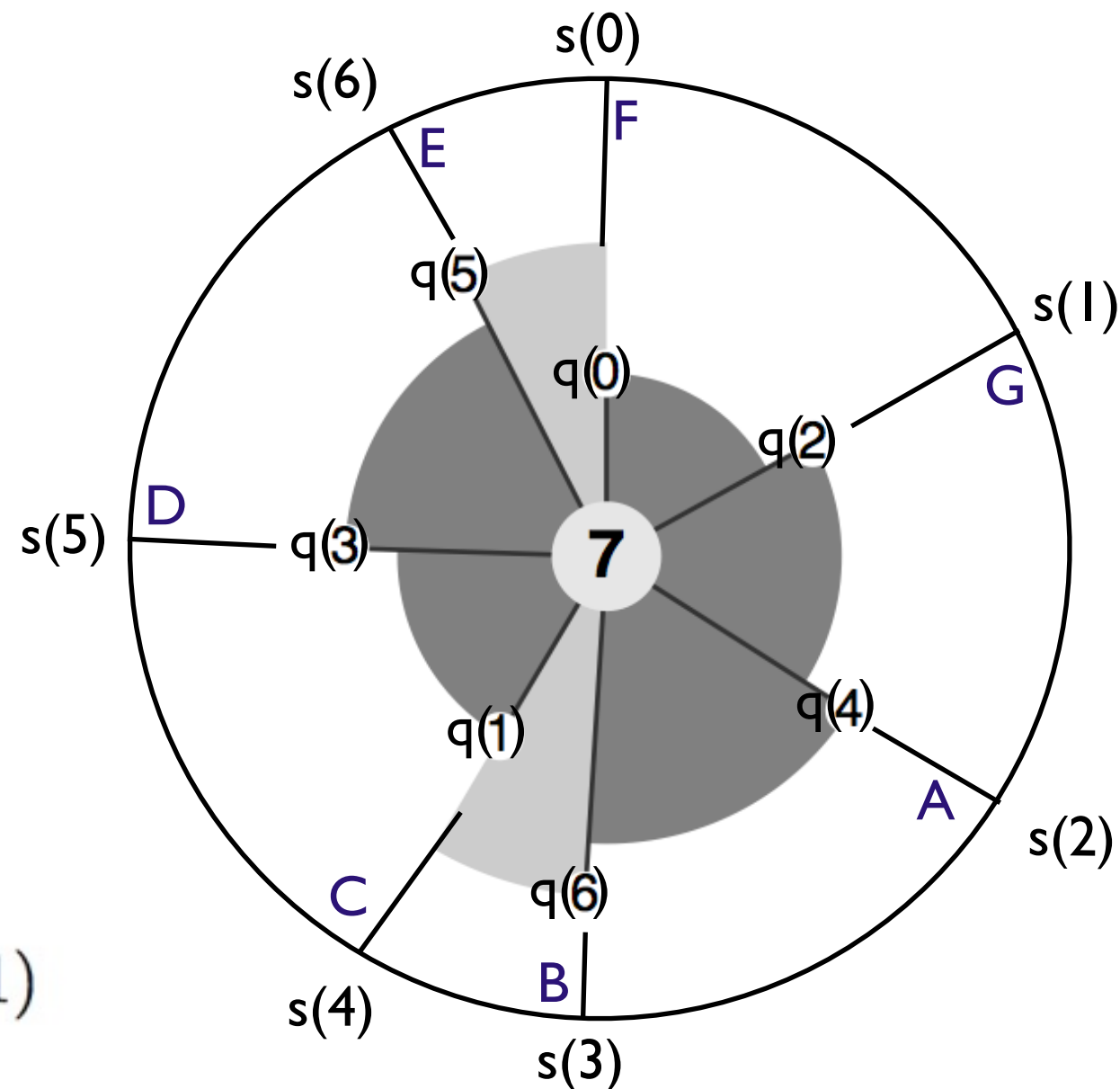
Generated Scale:

$$\begin{array}{ccc} \mathbb{Z}_n & & \\ p \downarrow \wr & \searrow \wr q & \\ \mathbb{Z}_n & \xrightarrow[\quad s \quad]{} & S \subset \mathbb{R}/\mathbb{Z} \end{array}$$

$$q(k) = g \cdot k \bmod 1 \text{ for some } g \in (0, 1)$$

Well-formed Scale:

The permutation $p = s^{-1}q : \mathbb{Z}_n \xrightarrow{\sim} \mathbb{Z}_n$ is a linear map.



Theorem (Carey & Clampitt 1989, 1996):

Consider an n -tone scale $S = \{k \cdot g \bmod 1 \mid k = 0, \dots, n-1\} \subset \mathbb{R}/\mathbb{Z}$, generated by $g \in (0, 1)$. Let $q, s : \mathbb{Z}_n \rightarrow S$ denote the associated generation order and scalar order encodings of S , respectively. The following three properties are equivalent:

- (i) S is non-degenerate well-formed, i.e. $s^{-1}(q(k)) = m \cdot k \bmod n$ for a suitable $m \in \mathbb{Z}_n$ and $S \neq \frac{1}{n}\mathbb{Z}/\mathbb{Z}$.
- (ii) (Myhill property) Each non-zero generic interval comes in precisely two specific sizes.
- (iii) The ratio $\frac{m}{n}$ is a semiconvergent of the generator g with $\frac{m}{n} \neq g$.

Example $g = \log_2(3/2)$:

with semiconvergents $1/2, 2/3, 3/5, 4/7, 7/12, \dots$



Is the automorphism
musically relevant?

$$\cdot m : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$$

$$\begin{array}{ccc} & \mathbb{Z}_n & \\ \cdot m \downarrow \wr & \searrow \wr & \\ \mathbb{Z}_n & \xrightarrow{\sim} & S \subset \mathbb{R}/\mathbb{Z} \end{array}$$

The diatonic case:

$$\begin{array}{l} n = 7 \\ m = 4 \end{array}$$

Every step can be divided into two fifths.
Every fifth can be divided into four steps.

$$4 \cdot 2 = 1 \pmod{7}$$

Excerpt from the madrigal “Questi vaghi”



Claudio Monteverdi

Diagram illustrating an excerpt from the madrigal “Questi vaghi” by Claudio Monteverdi, showing musical notation and structural analysis.

The excerpt is divided into measures 1 through 9, organized into two systems of staves (Treble and Bass).

Measure 1: Treble staff (labeled **a** in red), Bass staff (labeled **a** in red).

Measure 2: Treble staff (labeled **a'** in orange), Bass staff (labeled **a** in red).

Measure 3: Treble staff (labeled **a'** in orange), Bass staff (labeled **b** in red).

Measure 4: Treble staff (labeled **b'** in orange), Bass staff (labeled **b** in red).

Measure 5: Treble staff (labeled **c** in red), Bass staff (labeled **c** in red).

Measure 6: Treble staff (labeled **c'** in orange), Bass staff (labeled **c** in red).

Measure 7: Treble staff (labeled **c'** in orange), Bass staff (labeled **d** in red).

Measure 8: Treble staff (labeled **d'** in orange), Bass staff (labeled **d** in red).

Measure 9: Treble staff (labeled **d'** in orange), Bass staff (labeled **d** in red).

Blue lines (solid and dashed) indicate melodic contours and phrasing across the staves.

Beginning of the “Passacaille” (Suite for Harpsichord in G-Minor)



Georg Friedrich Händel



The image displays the beginning of the "Passacaille" from the Suite for Harpsichord in G-Minor by Georg Friedrich Händel. The score is written for harpsichord in G-Minor (one flat) and common time (C). The first system shows the initial four measures. The second system, starting at measure 5, is divided into four measures. The first three measures of the second system are enclosed in solid red rectangular boxes, while the fourth measure is enclosed in a dashed red box. A blue line with arrows indicates a sequence of chords or intervals across the measures: a solid line from the first measure to the second, a dashed line from the second to the third, and a solid line from the third to the fourth. The notation includes treble and bass staves with various musical symbols such as notes, rests, and accidentals.

1st Theme from a Piano Sonata



W.A. Mozart

Allegro maestoso.



A
(Fundamental Bass)

D G

C F

B E

A

Excerpt from a Study for Piano

39 40

41 42

Ped. * *Ped.* *

43

cresc.

44

Ped. * *Ped.* *

45

f

Ped. * *Ped.* *

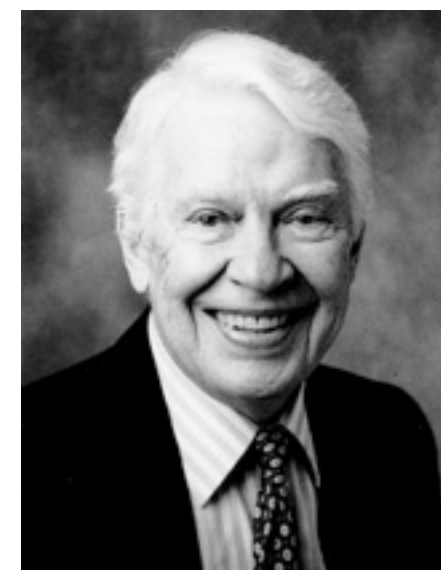


F. Chopin

Handwritten musical notation on two staves, featuring various chords and melodic lines. The notation is divided into four groups by red rectangular boxes:

- Group 1 (Staff 1): A_{mi}^7 and D_{mi}^7
- Group 2 (Staff 1): G^9 and C_{MA}^7
- Group 3 (Staff 2): F_{MA}^7 and $B_{mi}^7(b5)$
- Group 4 (Staff 2): $E^7(b9)$ and A_{mi}^7

Additional handwritten notes include $C\#O^7$ on the right of the second staff and a small mp marking under the first staff.



Bart Howard

Extending the Linear Automorphism

Octave #

Fifth Fourth

$$0 \rightarrow \mathbb{Z} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \end{pmatrix} \right] \hookrightarrow \mathbb{Z}[P5, P4] \xrightarrow{-} \mathbb{Z}_7 \rightarrow 0$$

$$\downarrow \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

$$\downarrow \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

$$\downarrow \cdot 4$$

$$0 \rightarrow \mathbb{Z} \left[\begin{pmatrix} 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] \hookrightarrow \mathbb{Z}[M2, m2] \xrightarrow{+} \mathbb{Z}_7 \rightarrow 0$$

Octave #

Major, minor
Seconds

let the words come into play !

Diatonic Modes

Ionian



Dorian



Phrygian



Lydian



Mixolydian



Aeolian



Locrian



Guidonian Modes



Glarean Modes



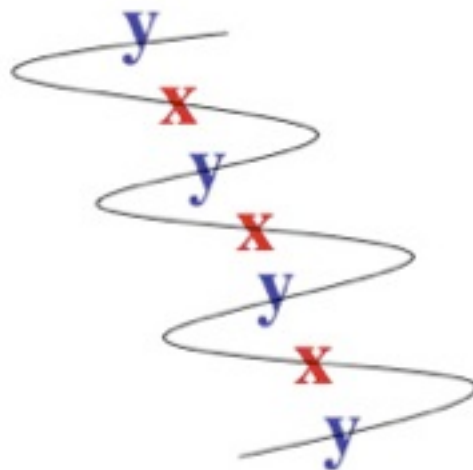
bad conjugate

Modes and their Plain Adjoints

Ionian		
Dorian		
Phrygian		
Lydian		
Mixolydian		
Aeolian		
Locrian		

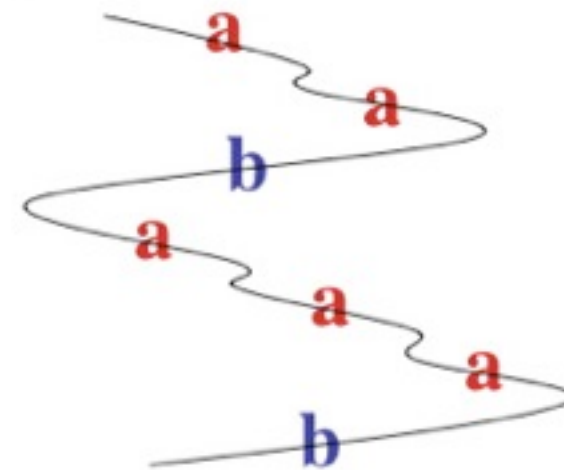
Calculation of the plain adjoint (by example)

a a b a | a a b
2 2 -5 2 | 2 2 -5
0 2 4 -1 1 3 5 (0)



y x | y x y x y

y x | y x y x y
-3 4 | -3 4 -3 4 -3
0 -3 1 -2 2 -1 3 (0)

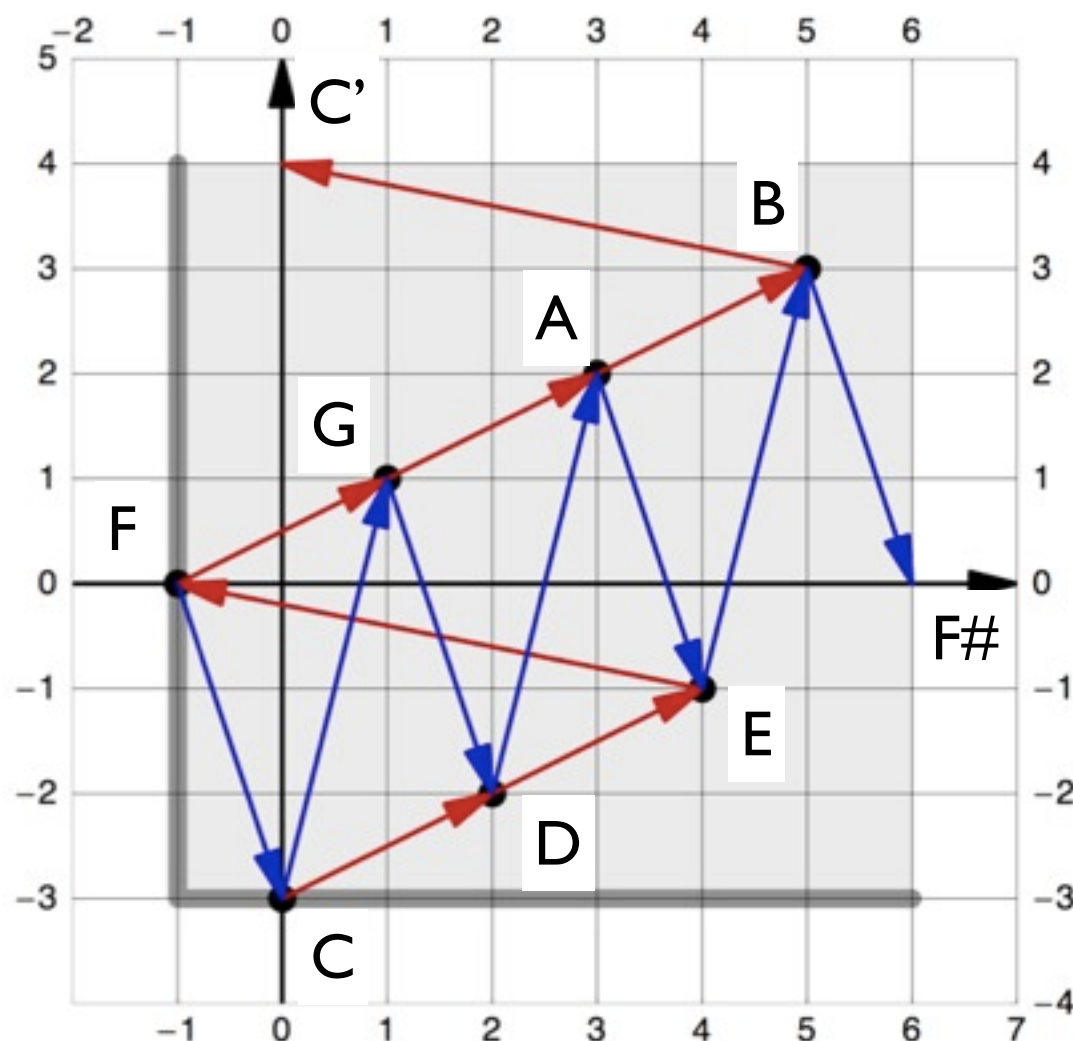


a a b a | a a b

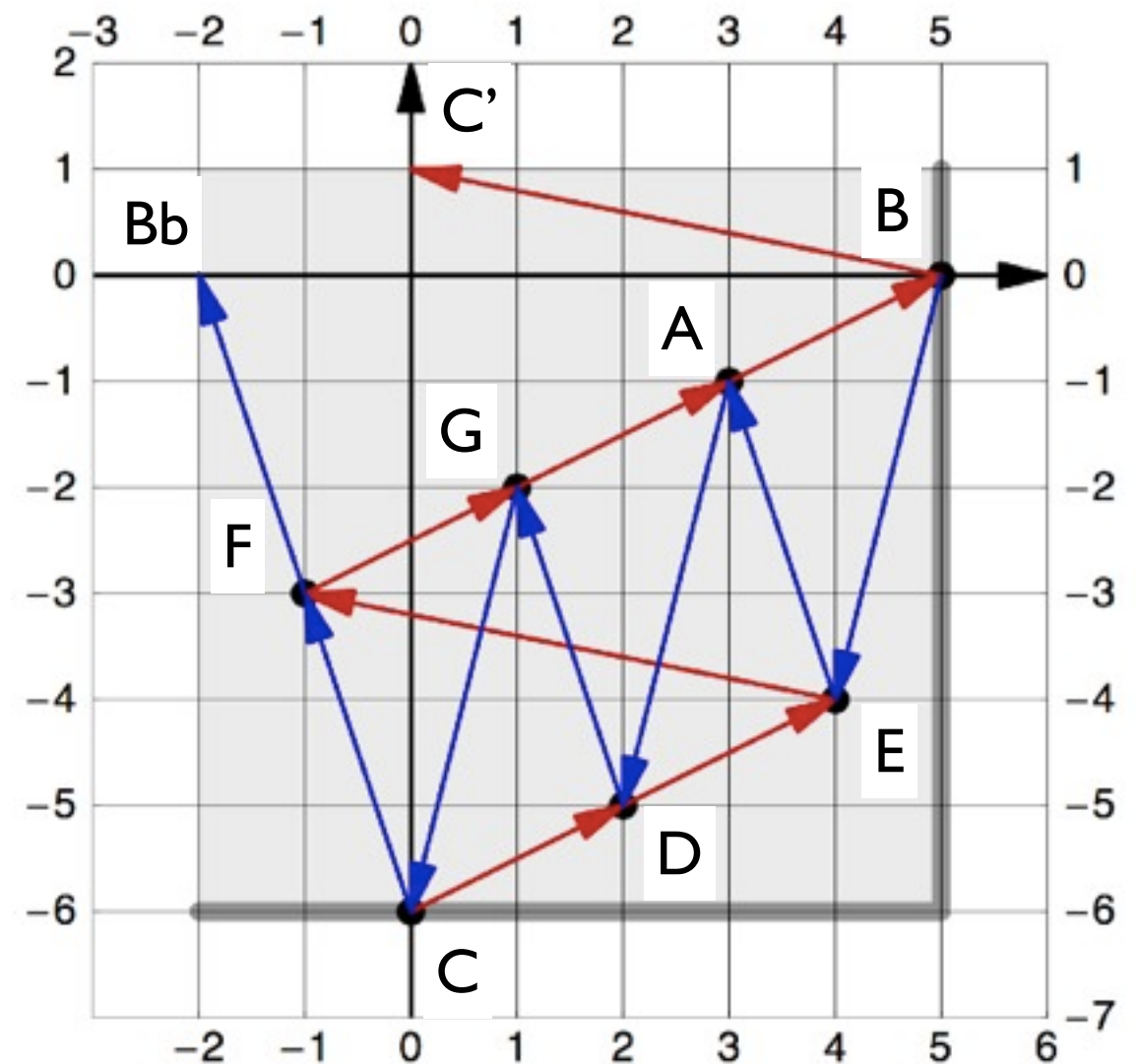


Comparing the two Adjoints in F_2

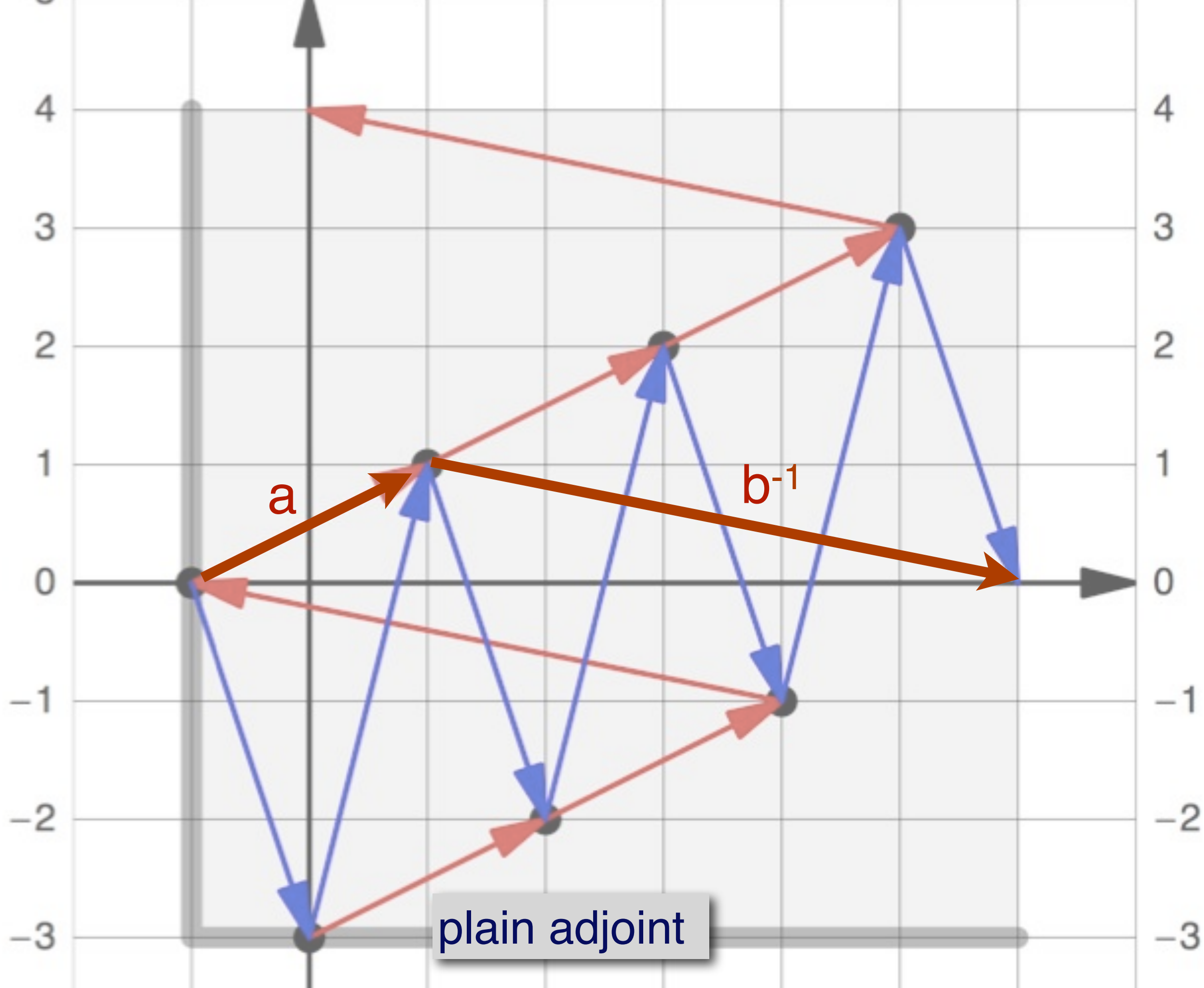
Automorphism f of F_2 : $x = f(a) = \text{aaba}$ $y = f(b) = \text{aab}$
 aabalaab $y^{-1}xly^{-1}xy^{-1}xy^{-1}$ aabalaab $x^{-1}ylyx^{-1}yx^{-1}yy$

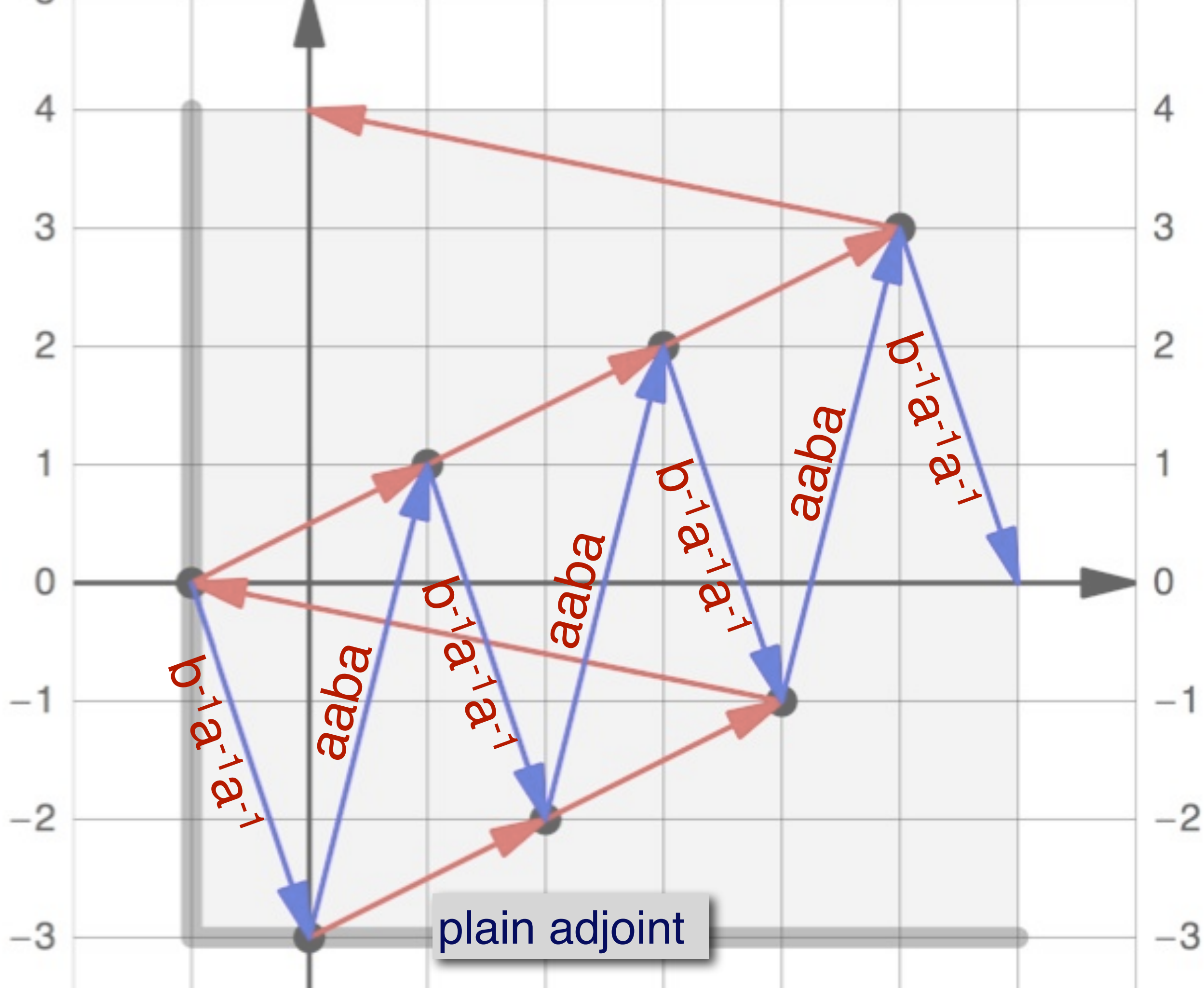


plain adjoint



twisted adjoint

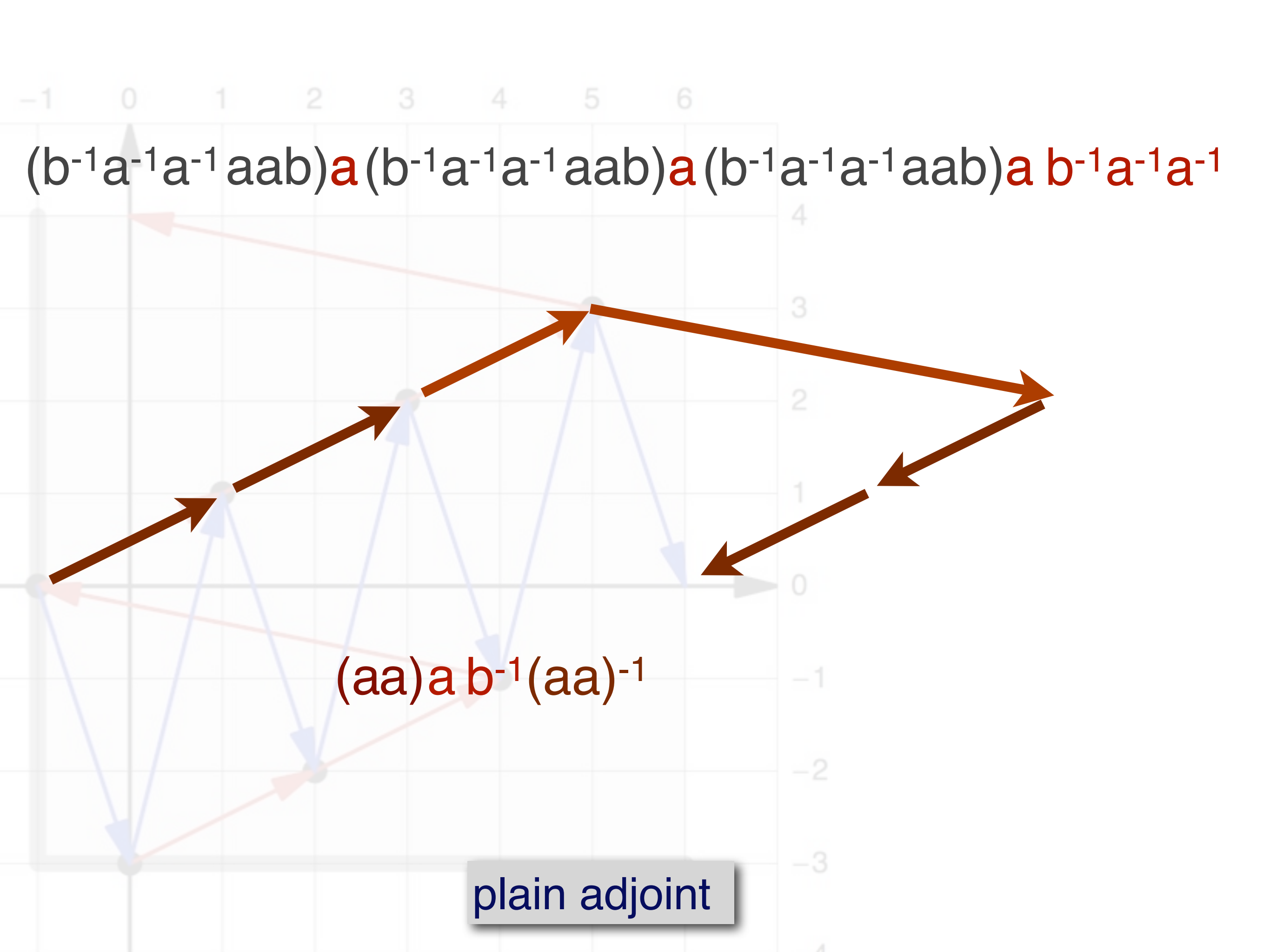


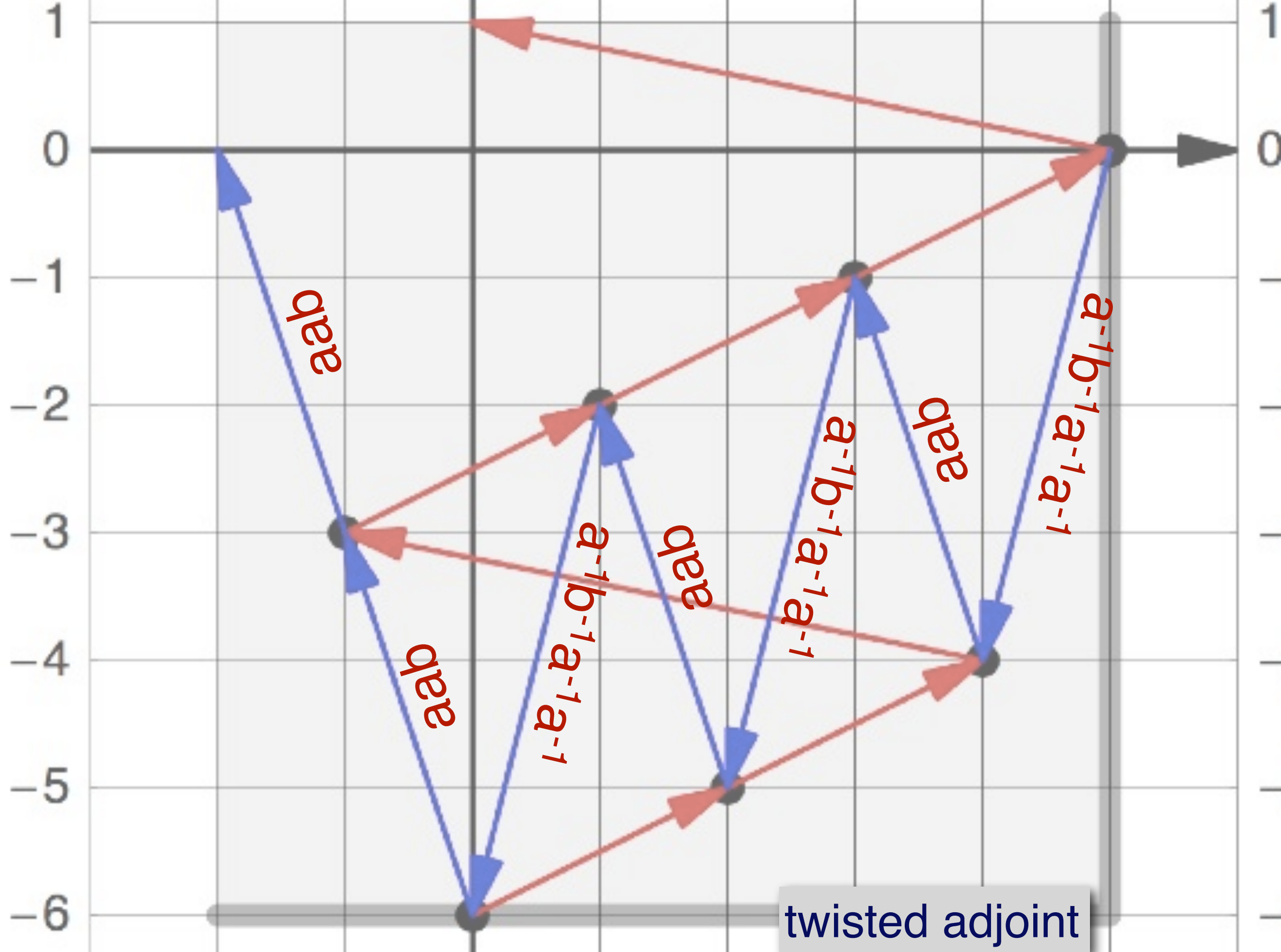


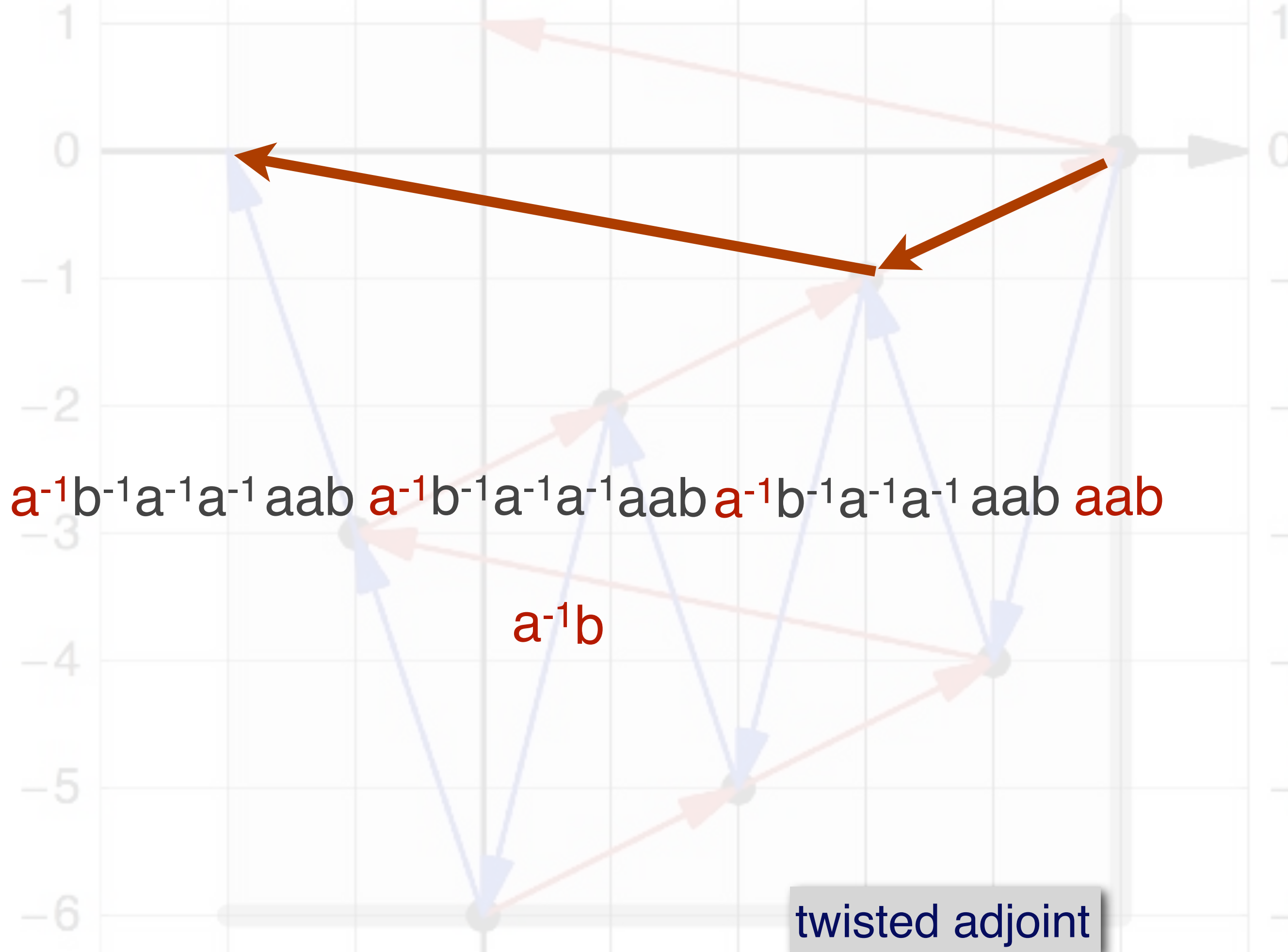
$$(b^{-1}a^{-1}a^{-1}aab)\mathbf{a}(b^{-1}a^{-1}a^{-1}aab)\mathbf{a}(b^{-1}a^{-1}a^{-1}aab)\mathbf{a}b^{-1}a^{-1}a^{-1}$$

$$(aa)\mathbf{a}b^{-1}(aa)^{-1}$$

plain adjoint







Reminder:

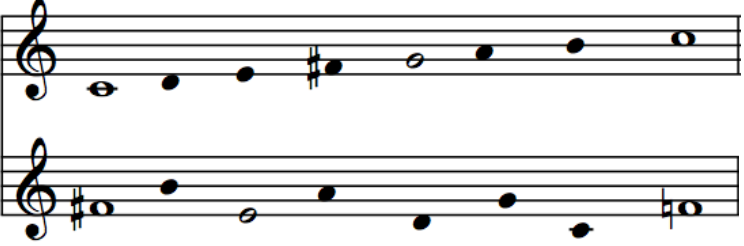
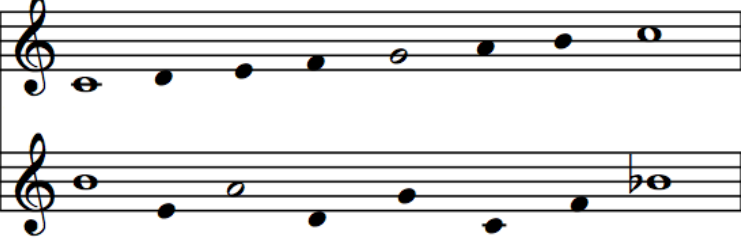
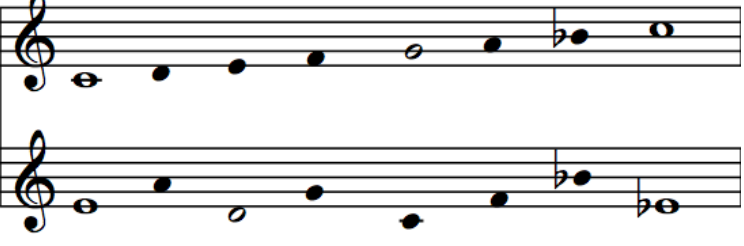
The group $\text{Aut}(F_2)$ is (redundantly) generated by the following automorphisms: $G, \tilde{G}, D, \tilde{D}, E : F_2 \rightarrow F_2$ with

$$\begin{aligned} G(a) &= a, & G(b) &= ab, & \tilde{G}(a) &= \widetilde{G(a)} = a, & \tilde{G}(b) &= \widetilde{G(b)} = ba, \\ D(a) &= ba & D(b) &= b, & \tilde{D}(a) &= \widetilde{D(a)} = ab, & \tilde{D}(b) &= \widetilde{D(b)} = b, \\ E(a) &= b, & E(b) &= a \end{aligned}$$

Proposition 4.1 *The mapping $f \mapsto f^* = (x^{-1}, y)f^{-1}(x^{-1}, y)$ is an involutive anti-automorphism of the special Sturmian monoid, that exchanges D and $\tilde{D}$ and fixes G and $\tilde{G}$. It sends conjugacy classes of morphisms onto conjugacy classes. The involution on Christoffel words that it induces is the same as the one of Section 2.*



Berthé, V., de Luca, A., Reutenauer, C.: On an involution of Christoffel words and Sturmian morphisms. *European Journal of Combinatorics* 29(2), 535–553 (2008)

$f_4 = GG\tilde{D} = (xxxy, xxy)$		$f_4^* = DGG = (yx, yxyxy)$
$f_1 = GGD = (xxyx, xxy)$		$f_1^* = \tilde{D}GG = (xy, xyxyy)$
$f_5 = G\tilde{G}\tilde{D} = (xxyx, xyx)$		$f_5^* = DG\tilde{G} = (yx, yxyyx)$
$f_2 = \tilde{G}GD = (xyxx, xyx)$		$f_2^* = \tilde{D}G\tilde{G} = (xy, xyxyx)$
$f_6 = \tilde{G}\tilde{G}\tilde{D} = (xyxx, yxx)$		$f_6^* = D\tilde{G}\tilde{G} = (yx, yyxyx)$
$f_3 = \tilde{G}\tilde{G}D = (yxxx, yxx)$		$f_3^* = \tilde{D}\tilde{G}\tilde{G} = (xy, yxyxy)$

Question:

The morphisms nicely generate the interval patterns.
But we don't have access to the **notes** yet.
Is there a **transformational** approach?

The Answer is here:

Berthé, V., de Luca, A., Reutenauer, C.: On an involution of Christoffel words and Sturmian morphisms. *European Journal of Combinatorics* 29(2), 535–553 (2008)

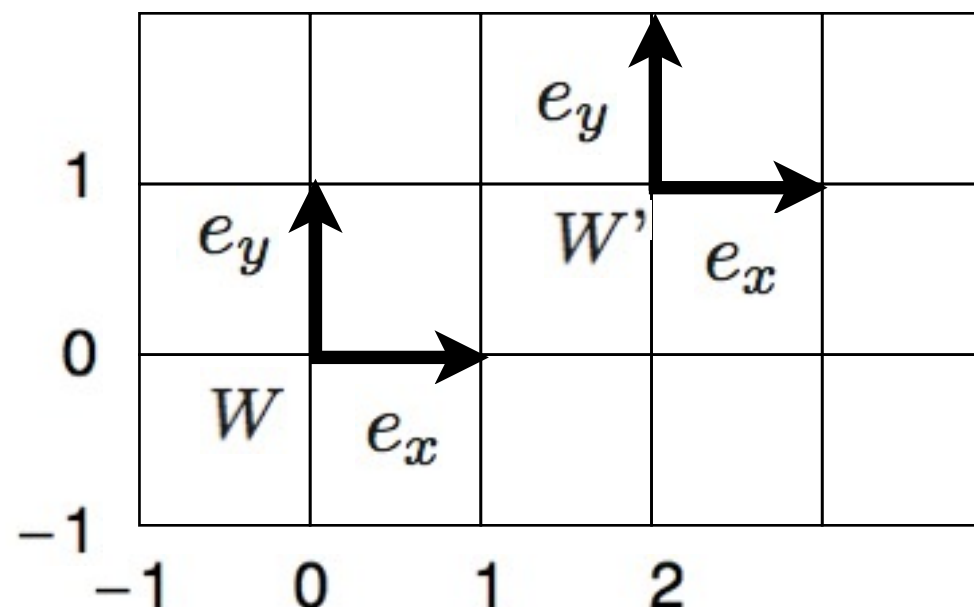
Arnoux, P., Shunji, I.: Pisot substitutions and Rauzy fractals. *Bulletin of the Belgian Mathematical Society Simon Stevin* 8, 181–207 (2001)



Sturmian Morphisms generate Lattice-path Transformations

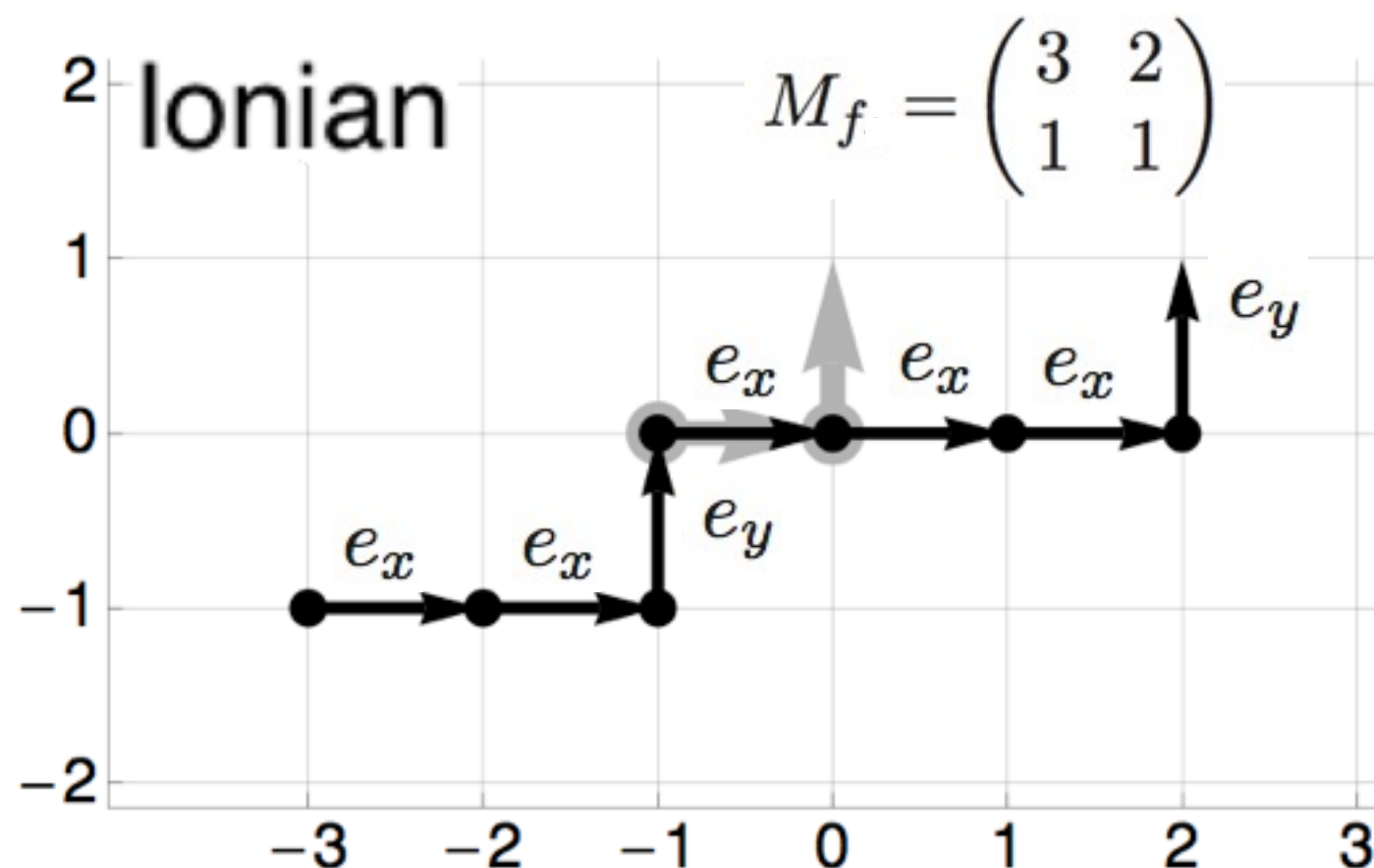
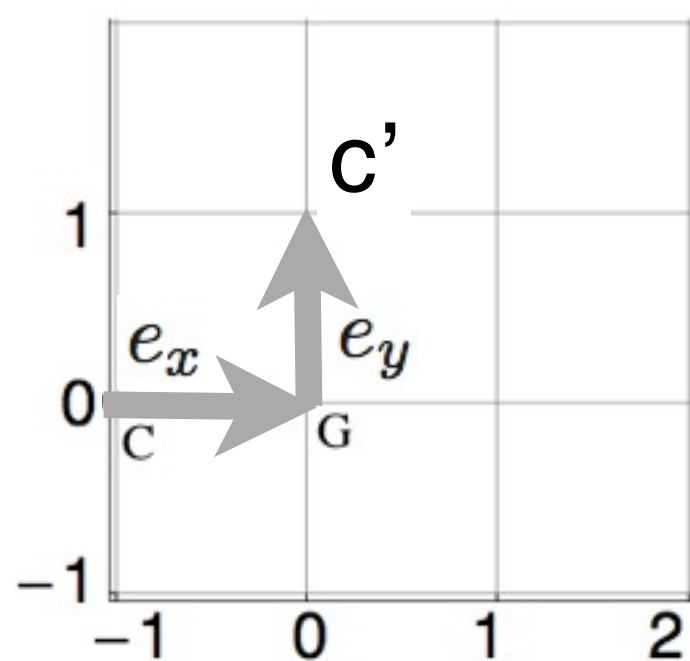
Let $e_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = V(x)$ and $e_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = V(y)$ denote the base vectors of $\mathbb{Z}^2$. Consider the set $\mathcal{B} = \{(W, e_x) \mid W \in \mathbb{Z}^2\} \sqcup \{(W, e_y) \mid W \in \mathbb{Z}^2\}$ and consider the linear space

$$\mathcal{F} = \{v : \mathcal{B} \rightarrow \mathbb{R} \mid v(W, e_z) = 0, \text{ for all but finitely many } (W, e_z) \in \mathcal{B}\} .$$



Sturmian Morphisms generate Lattice-path Transformations

$$E(f)(W, e_z) := \sum_{k=1}^{|f(z)|} (M_f \cdot W + V(P_k(f(z))), e_{L_k(f(z))})$$



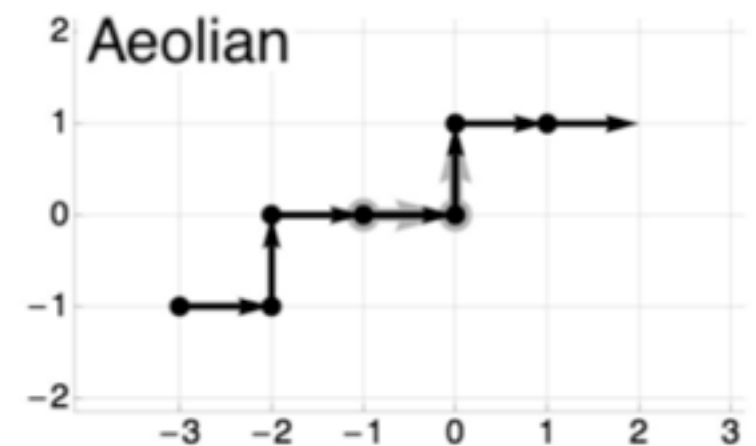
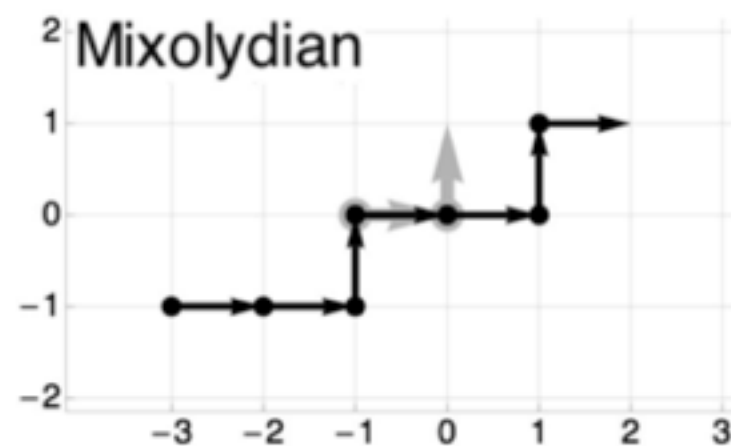
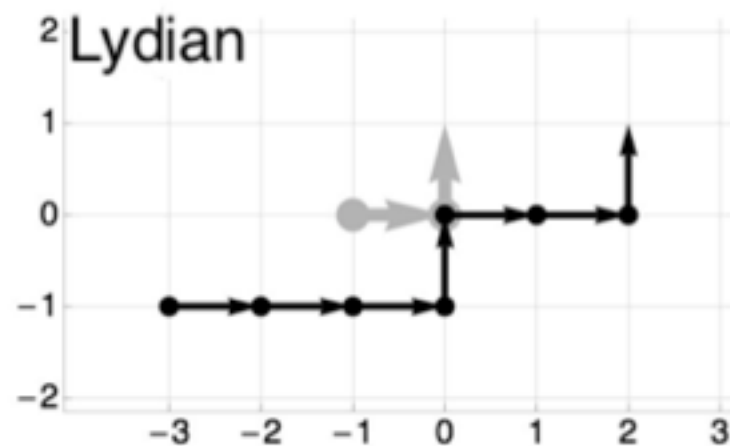
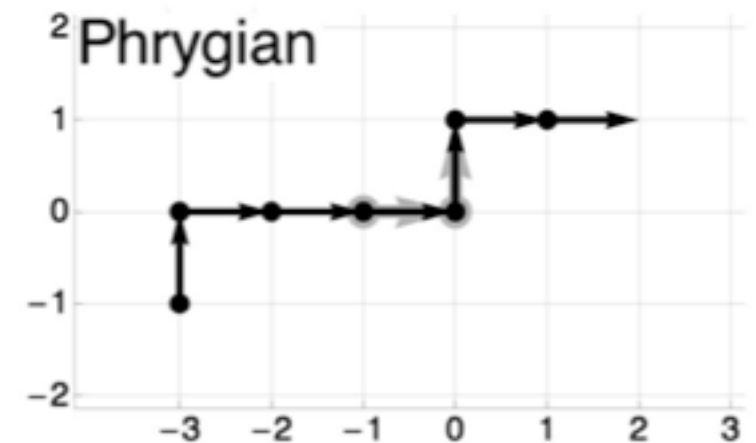
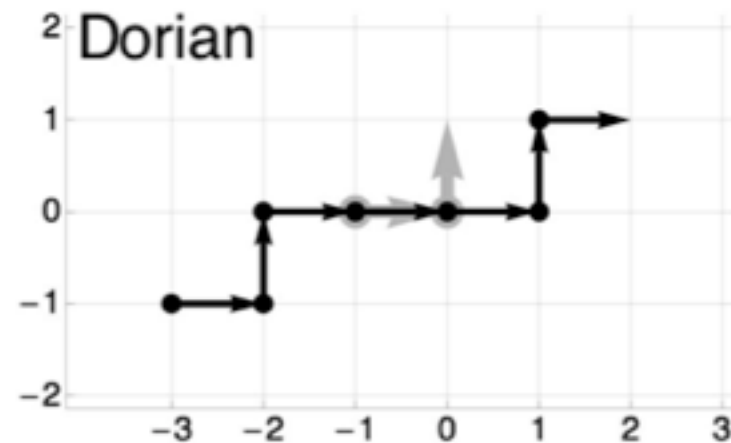
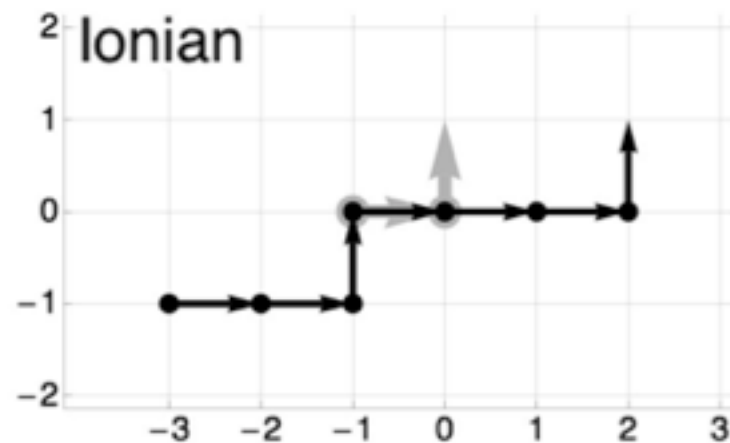
Common Finalis Modes (“Tropes”):

The lattice-path transformations are applied to the same initial lattice path.

$$f_1 = (xyx, xxy)$$

$$f_2 = (xyxx, xyx)$$

$$f_3 = (yxxx, yxx)$$



$$f_4 = (xxxxy, xxy)$$

$$f_5 = (xxyx, xyx)$$

$$f_6 = (xyxx, yxx)$$

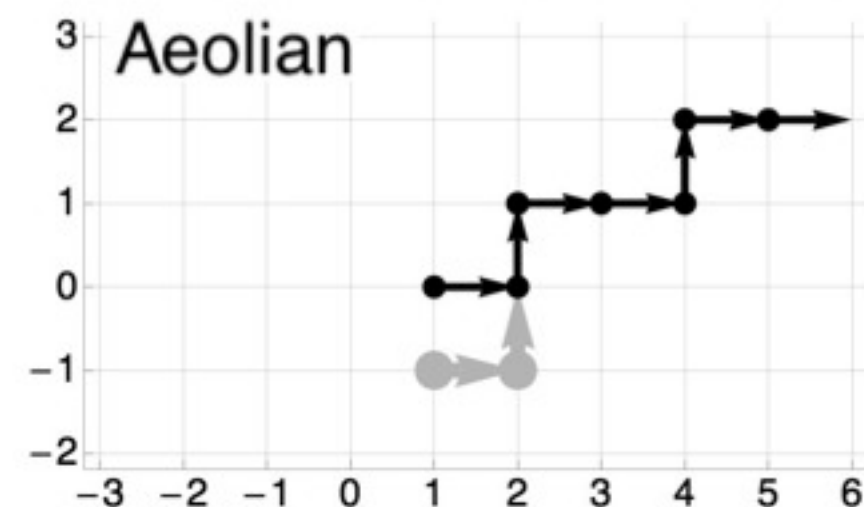
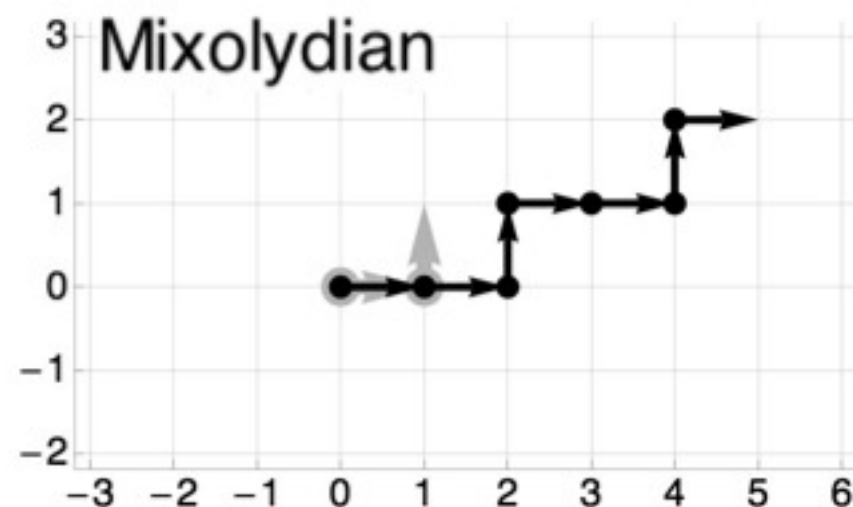
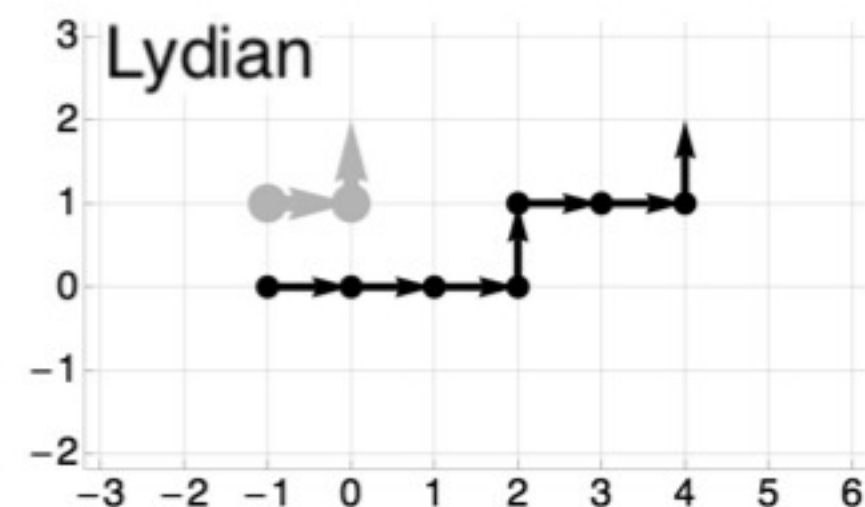
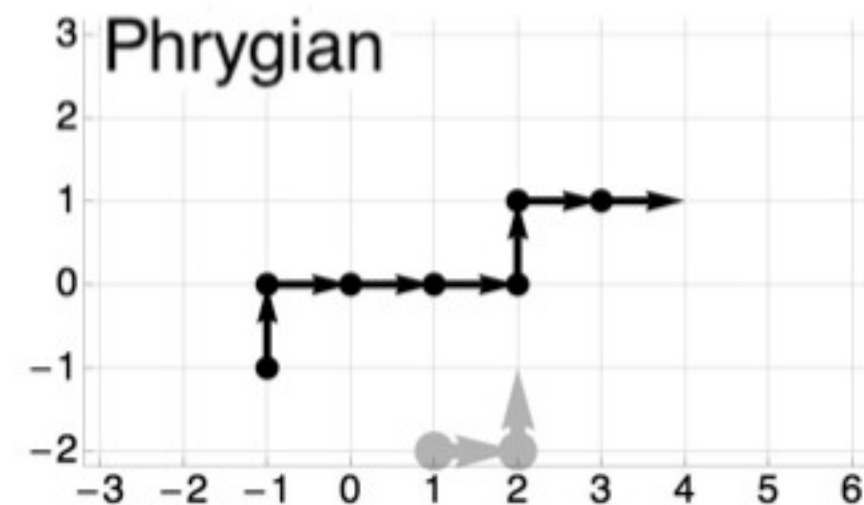
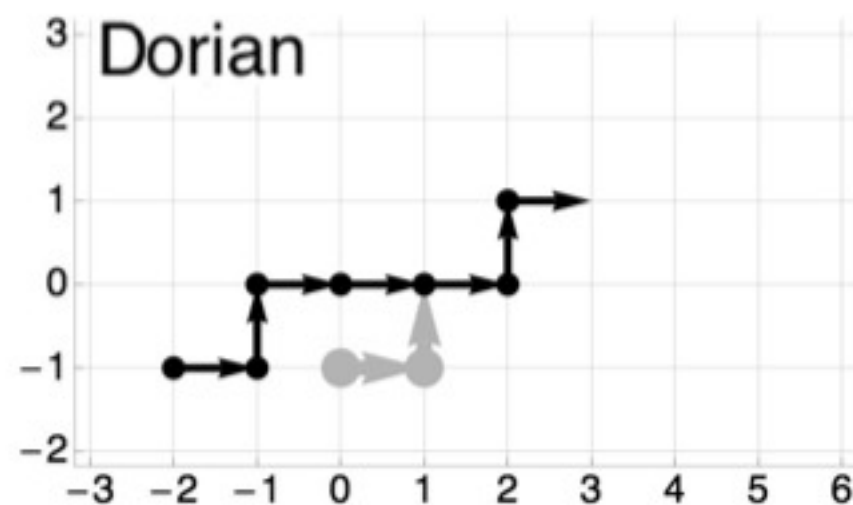
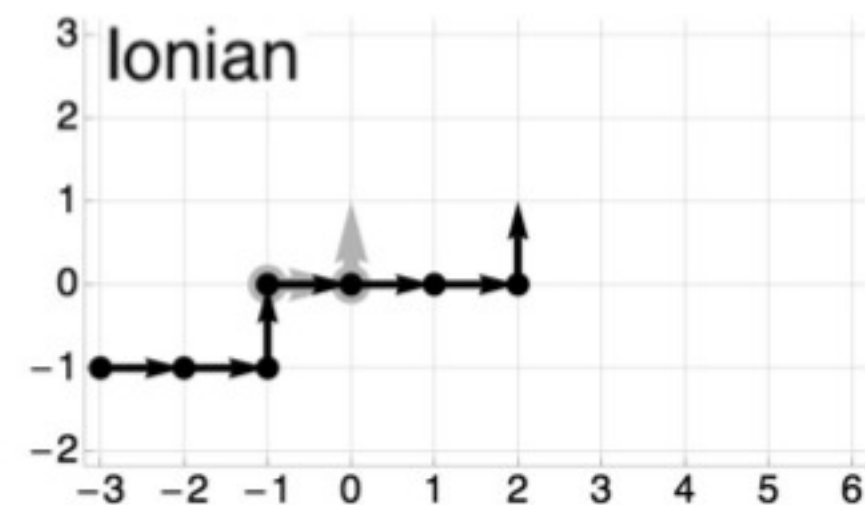
Common Origin (“White Note”) Modes:

The lattice-path transformations are applied to the different initial lattice paths.

$$f_1 = (xxyx, xxy)$$

$$f_2 = (xyxx, xyx)$$

$$f_3 = (yxxx, yxx)$$



$$f_4 = (xxxx, xxy)$$

$$f_5 = (xxyx, xyx)$$

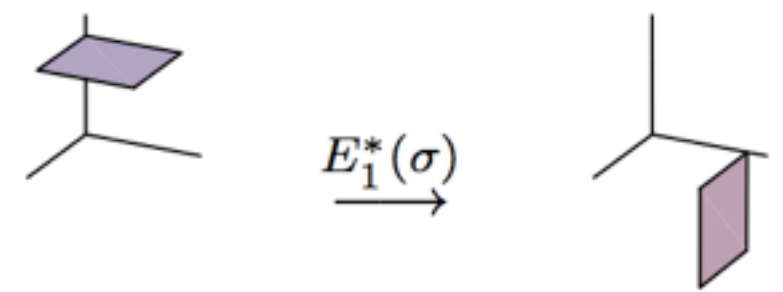
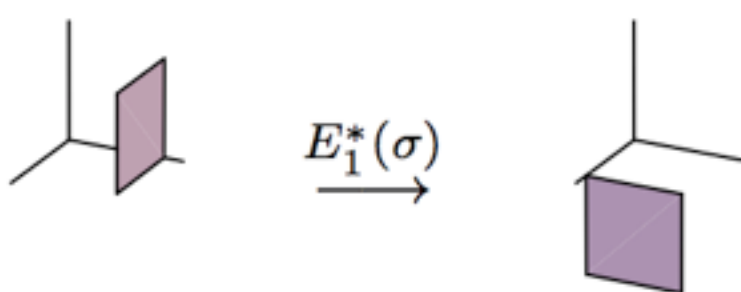
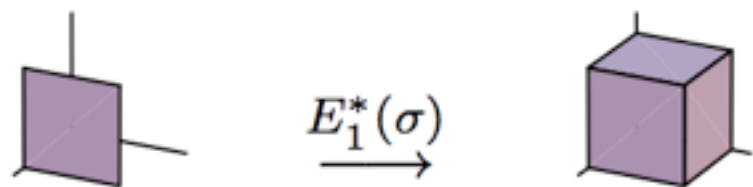
$$f_6 = (xyxx, yxx)$$

Lattice Path Transformations have Linear Adjoints

$$E(f)(W, e_z) := \sum_{k=1}^{|f(z)|} (M_f \cdot W + V(P_k(f(z))), e_{L_k(f(z))})$$

$$\begin{aligned} E(f)^*(W, e_z)^* = & \sum_{L_j(f(x))=z} (M_f^{-1}(W - V(P_j(f(x)))) , e_x)^* \\ & + \sum_{L_j(f(y))=z} (M_f^{-1}(W - V(P_j(f(y)))) , e_y)^* \end{aligned}$$

Geometric Interpretation (here in 3 Dimensions: Example from Arnoux and Ito)



Lattice Path Transformations have Linear Adjoints

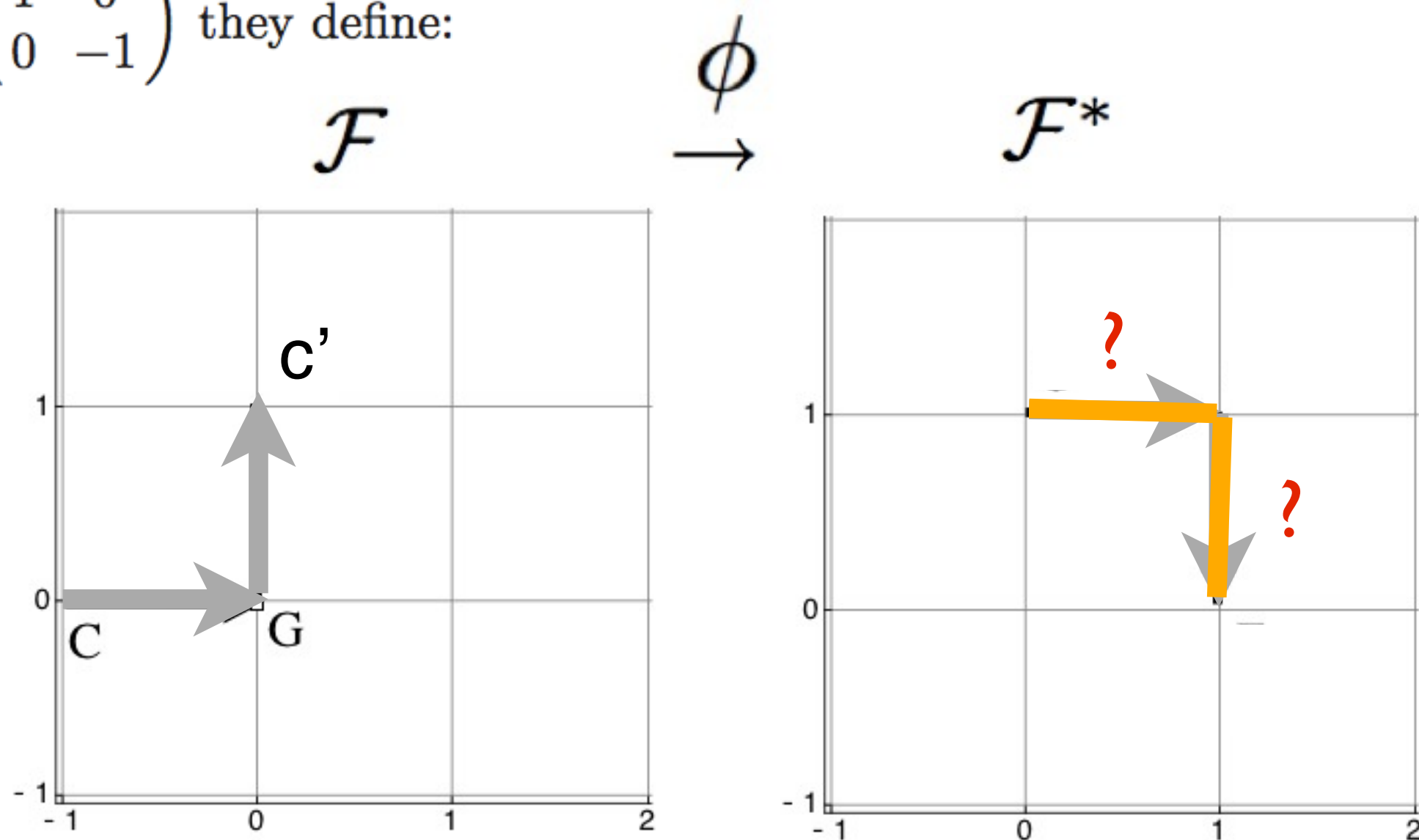
$$E(f)(W, e_z) := \sum_{k=1}^{|f(z)|} (M_f \cdot W + V(P_k(f(z))), e_{L_k(f(z))})$$

$$\begin{aligned} E(f)^*(W, e_z)^* &= \sum_{L_j(f(x))=z} (M_f^{-1}(W - V(P_j(f(x)))) , e_x)^* \\ &+ \sum_{L_j(f(y))=z} (M_f^{-1}(W - V(P_j(f(y)))) , e_y)^* \end{aligned}$$

$$M_f^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

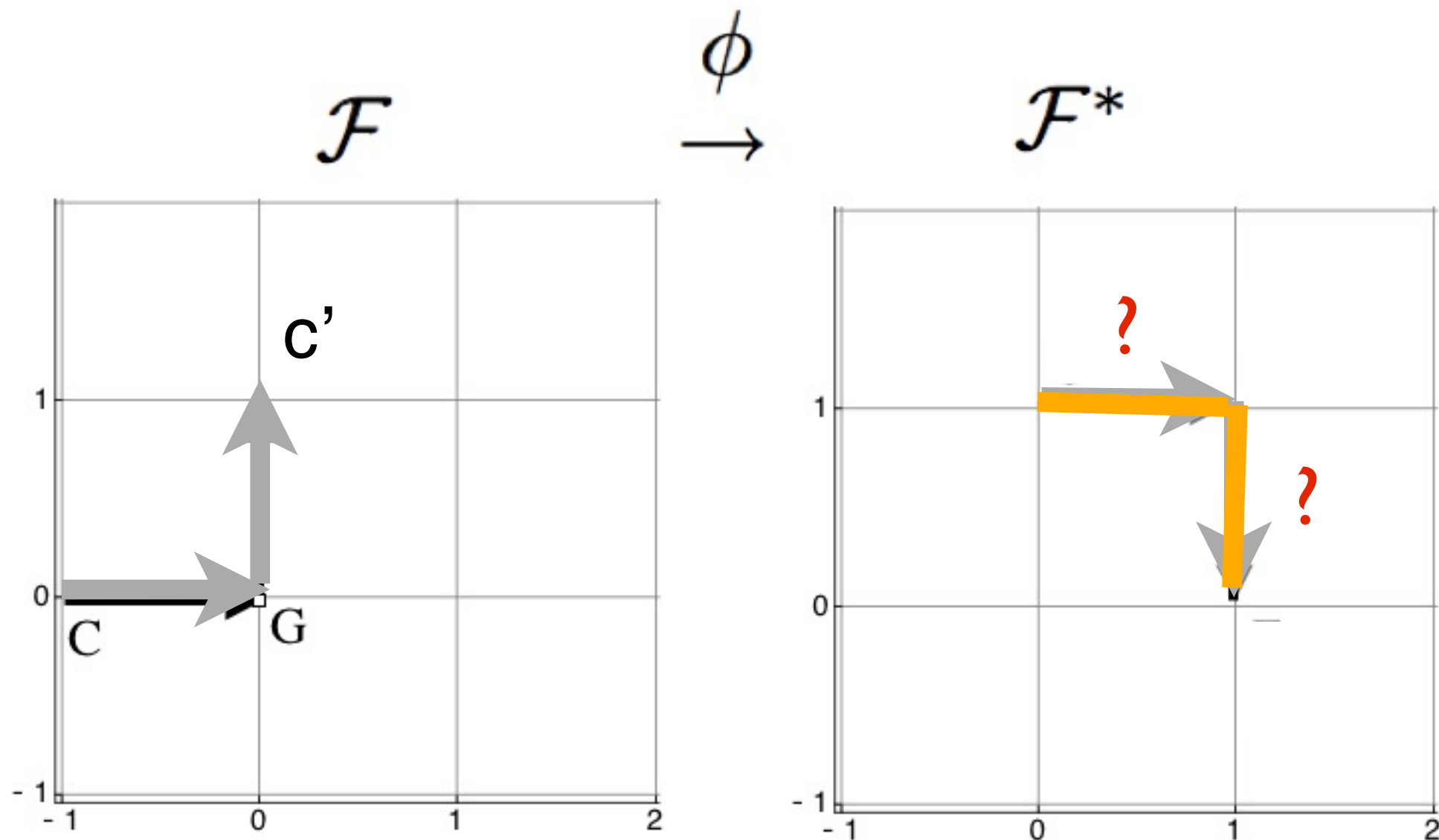
The initial lattice path can be mapped to the dual space

In order to establish a comparison between the maps $E(f)^* : \mathcal{F}^* \rightarrow \mathcal{F}^*$ and $E(f^*) : \mathcal{F} \rightarrow \mathcal{F}$ Berthe et al. [2] introduce the following map $\phi : \mathcal{F} \rightarrow \mathcal{F}^*$. With $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ they define:



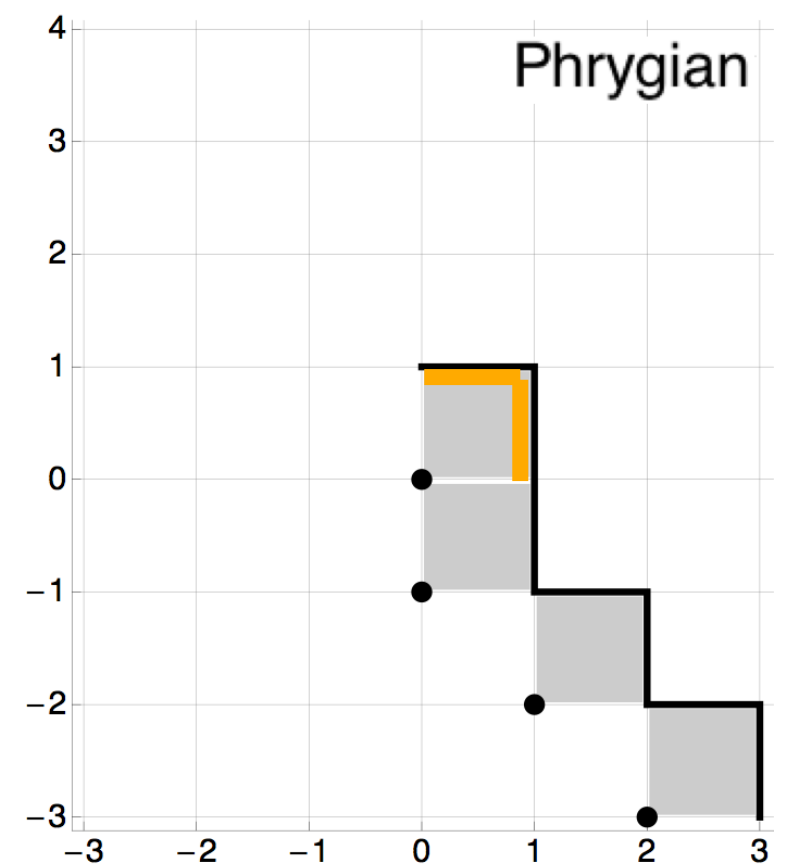
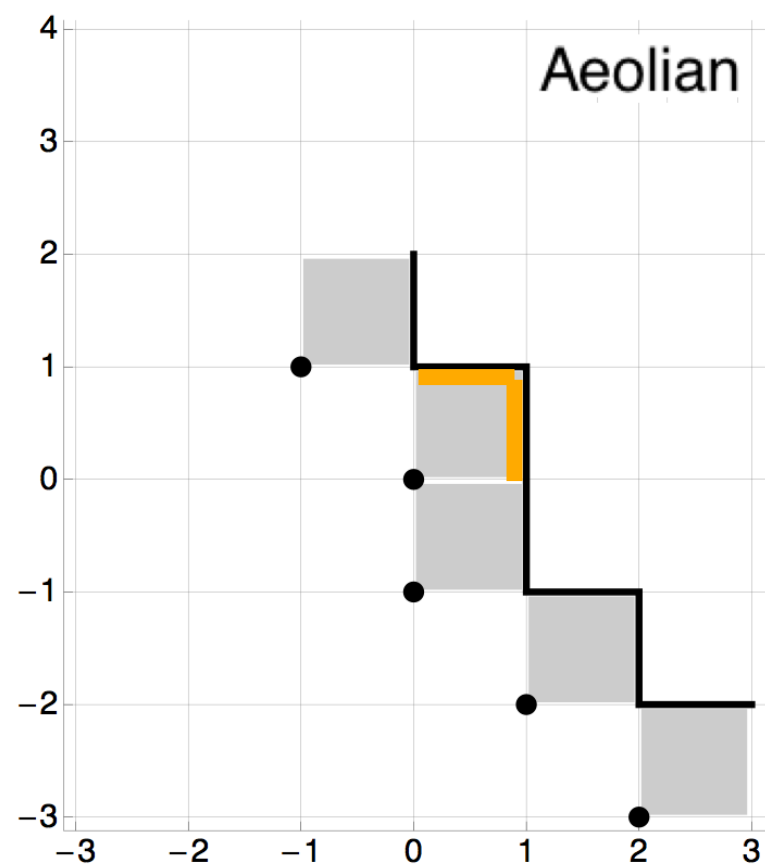
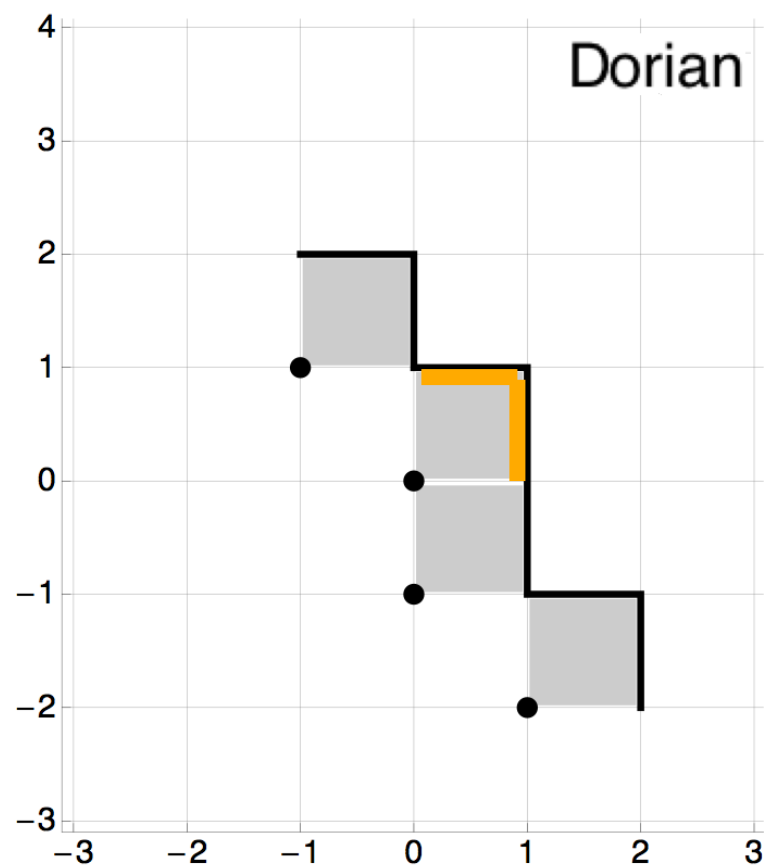
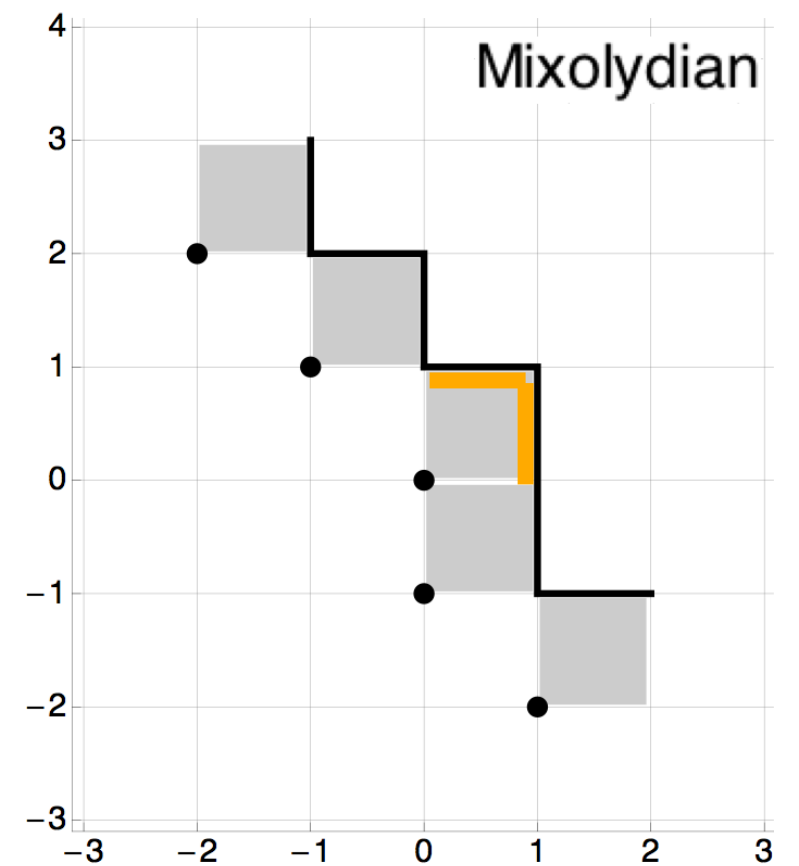
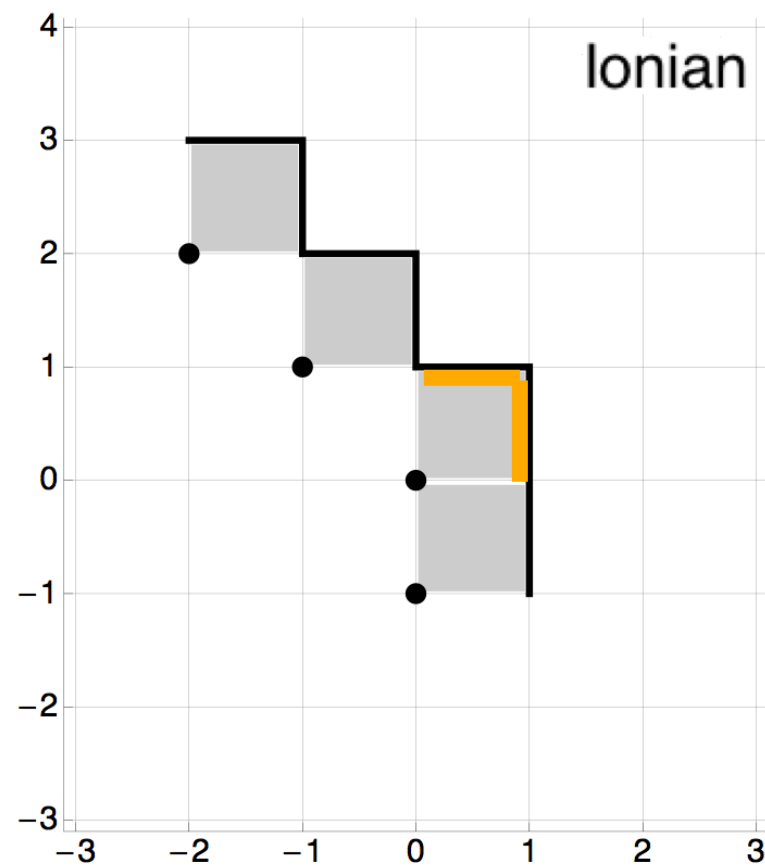
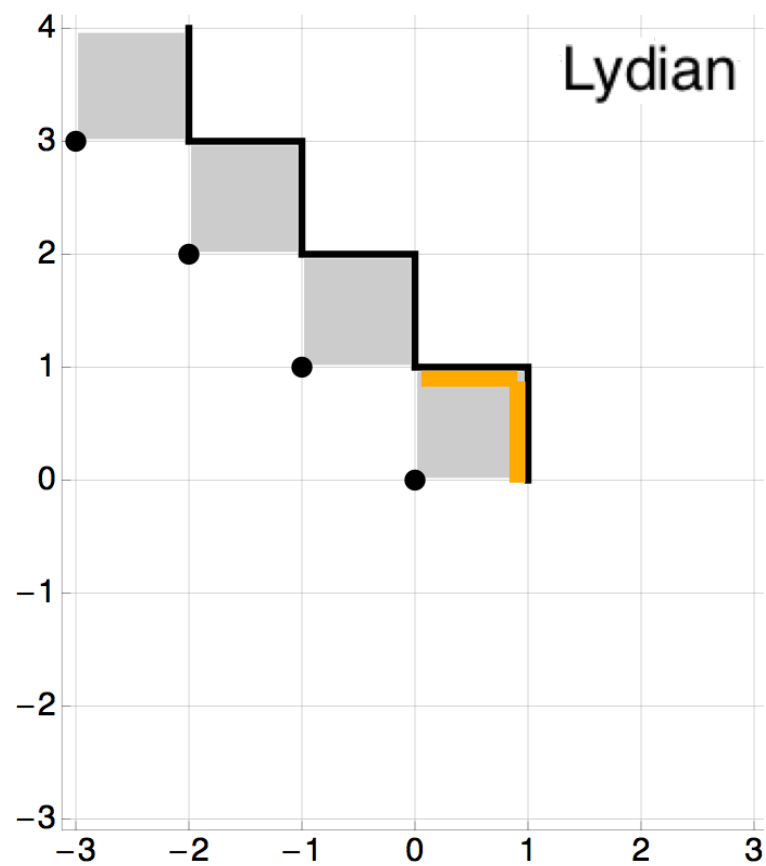
$$\phi(W, e_x) := (H \cdot W + e_x, e_y)^*, \quad \phi(W, e_y) := (H \cdot W, e_x)^* .$$

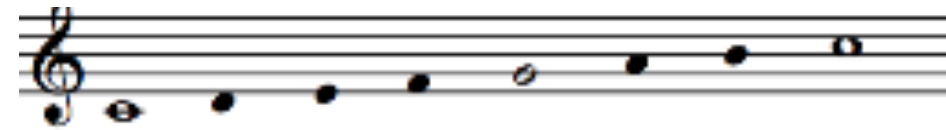
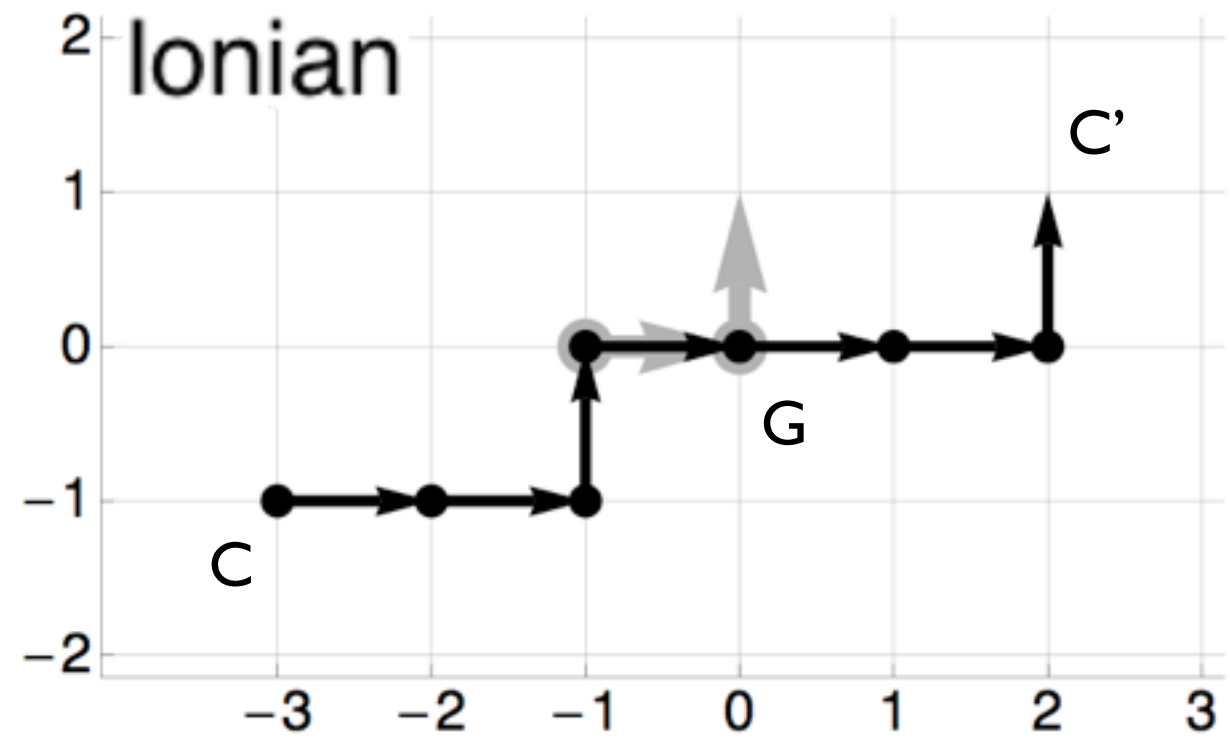
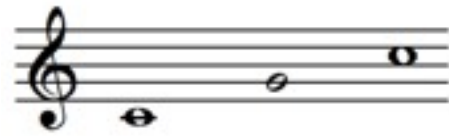
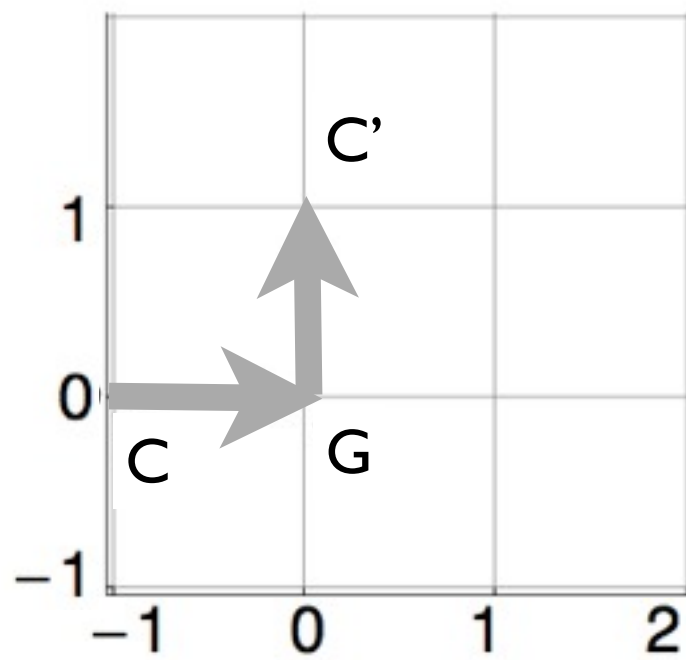
The initial lattice path can be mapped to the dual space



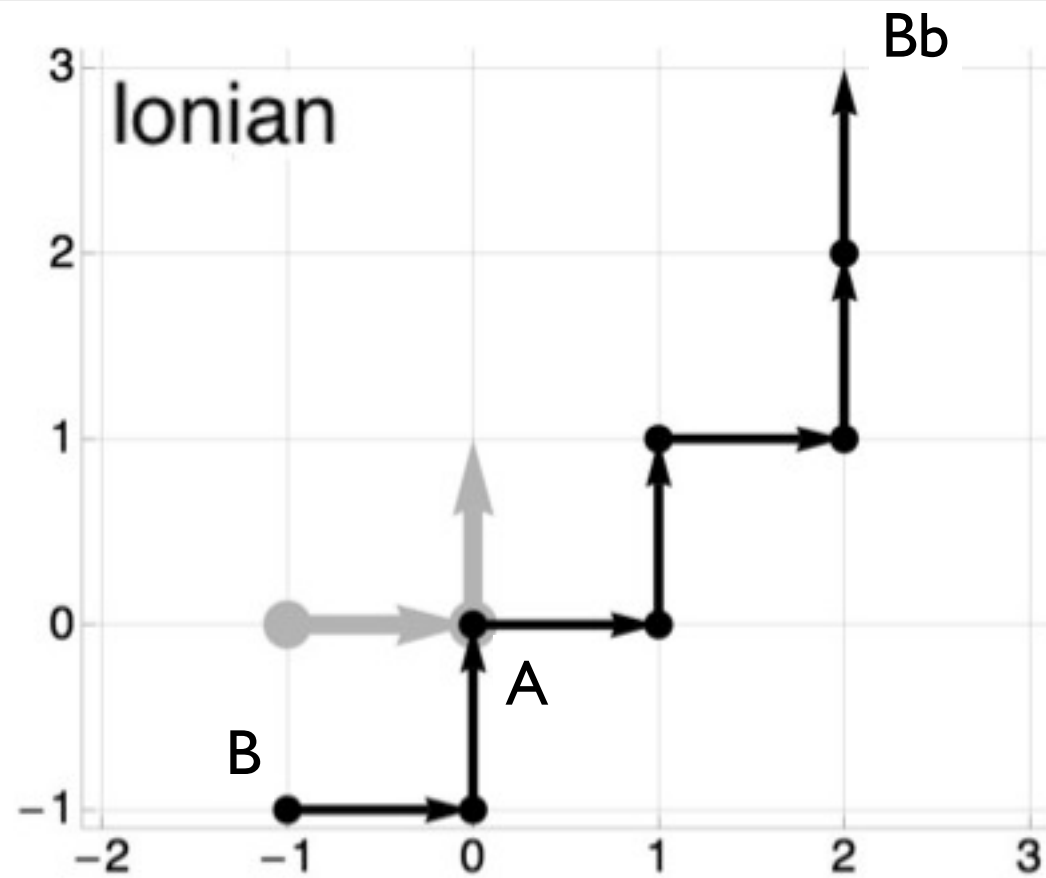
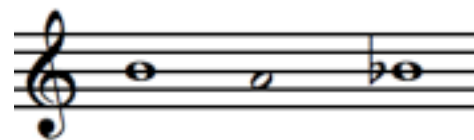
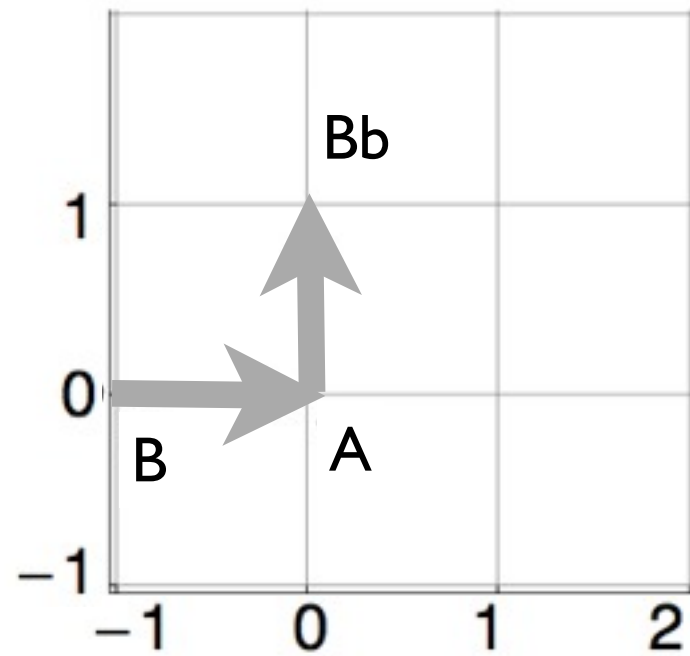
and we can apply the adjoints $E(f_i)^*$
of the 6 lattice path transformations $E(f_i)$...

... and obtain the associated co-vectors

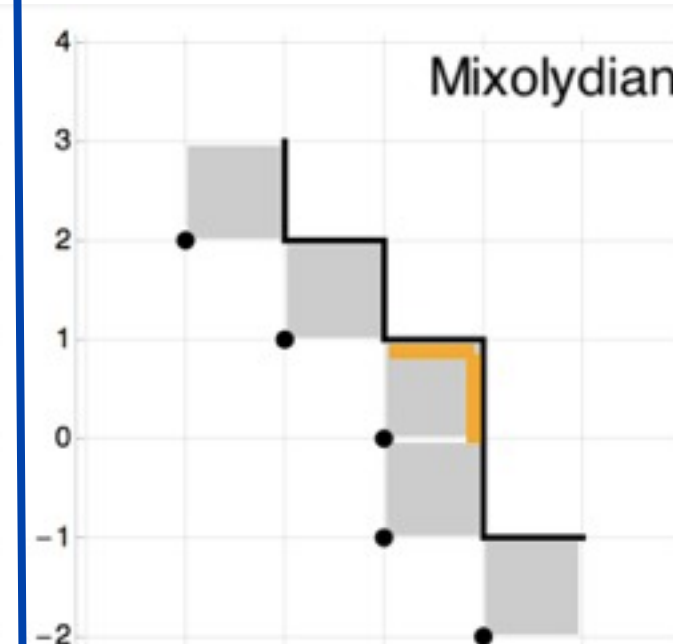
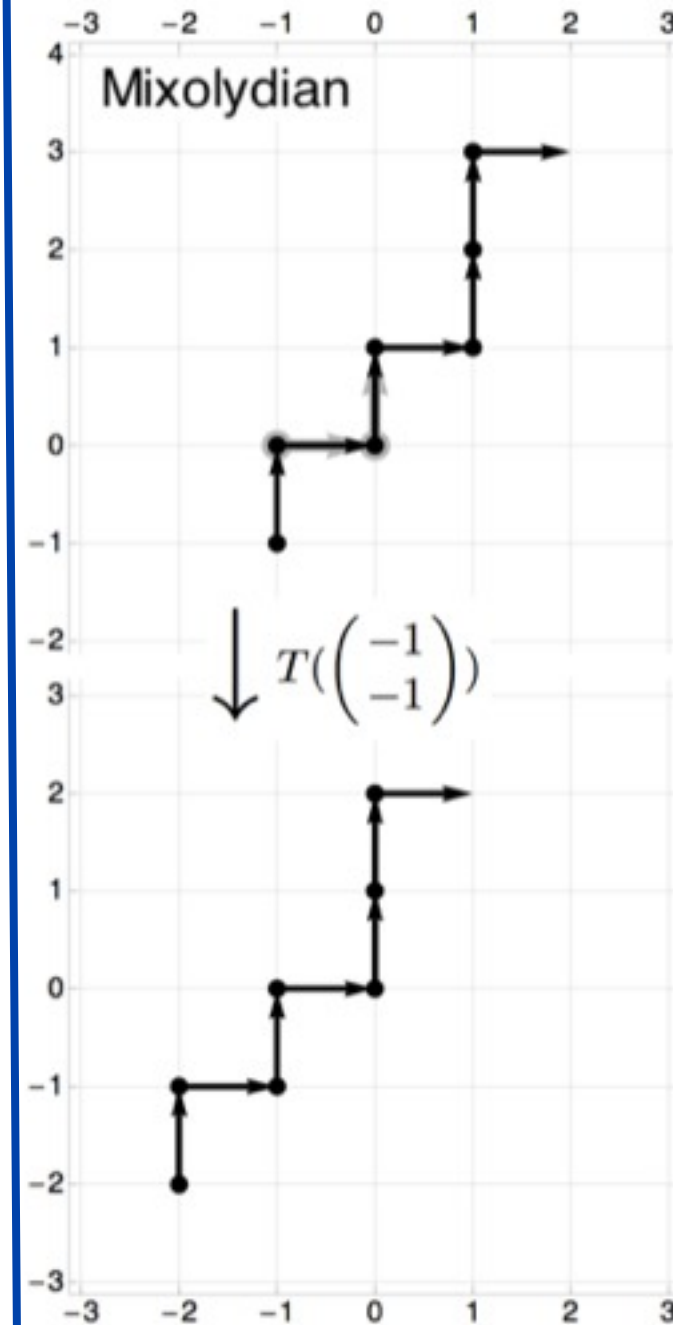
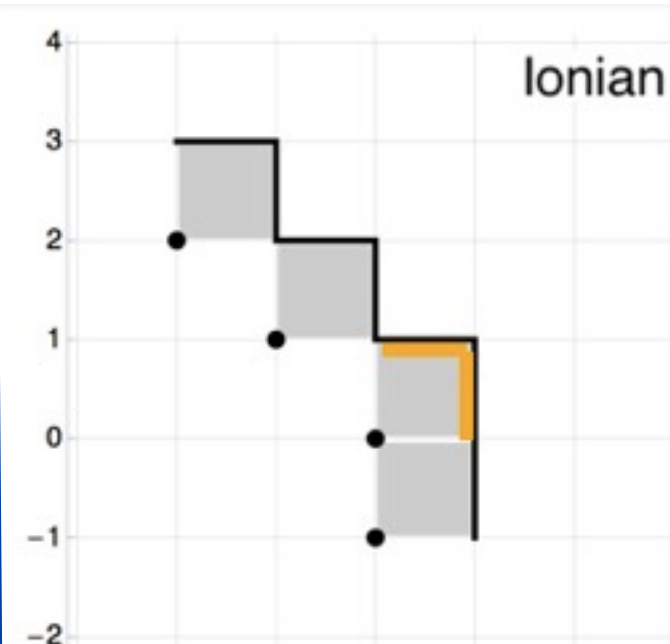
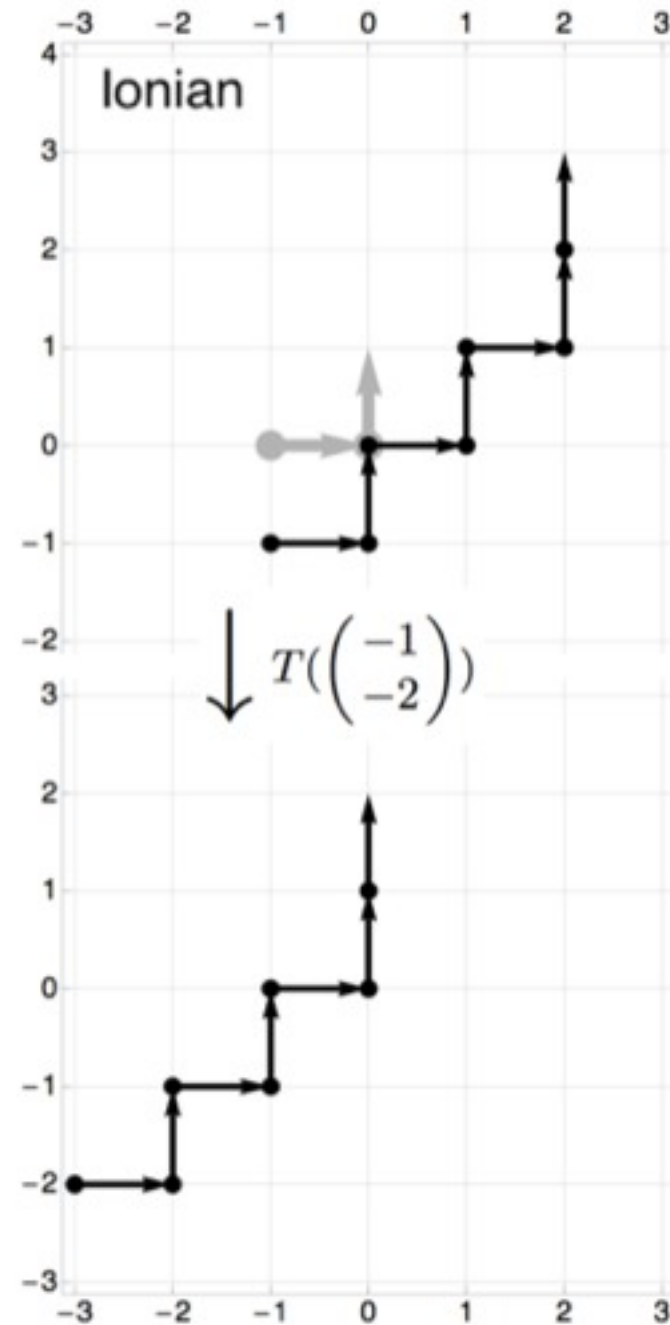
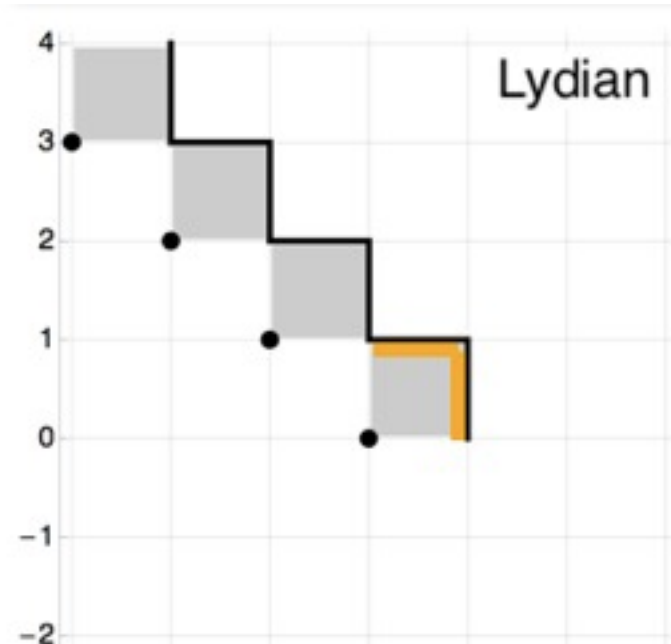
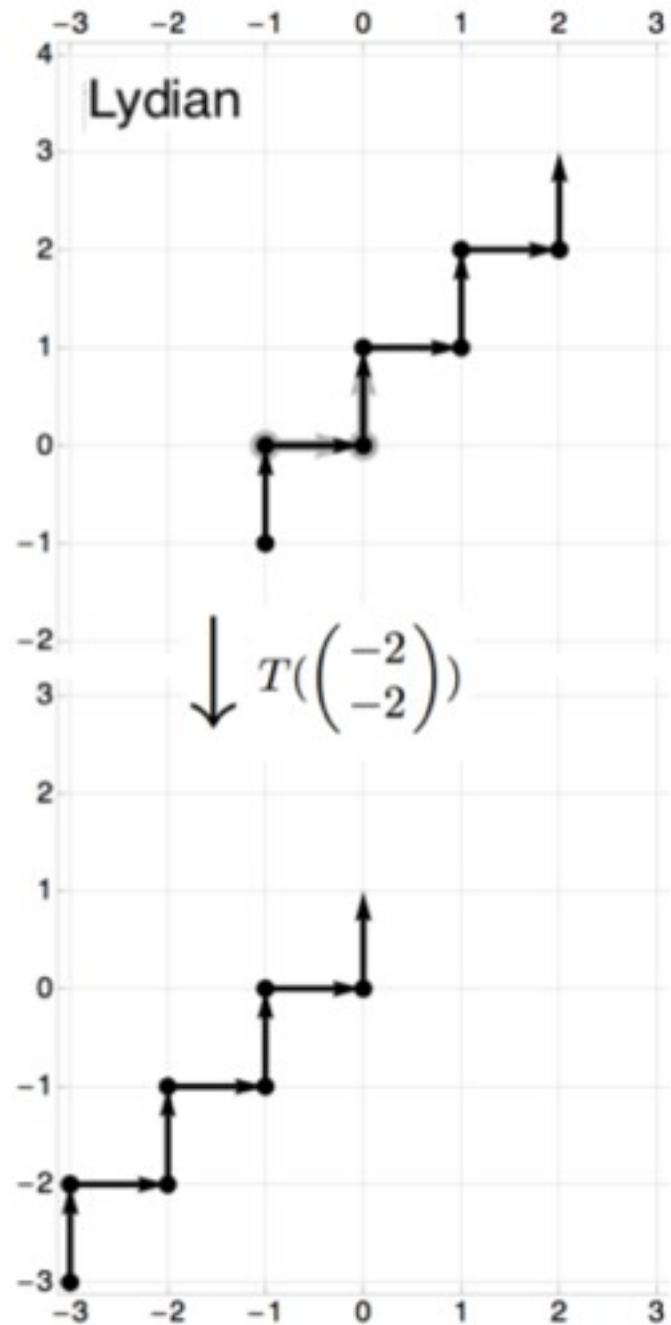


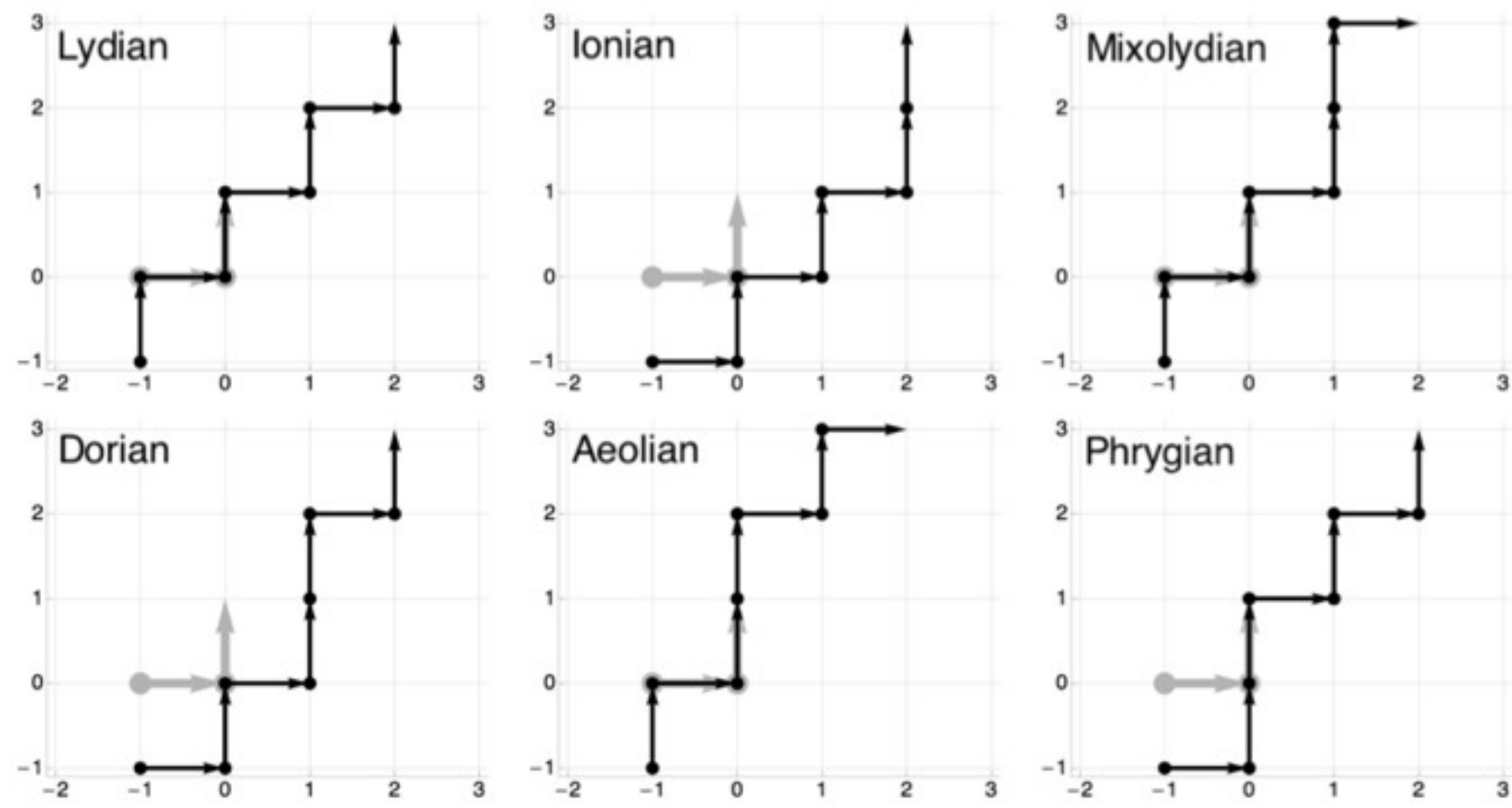


Switch music-theoretical interpretation



ϕ

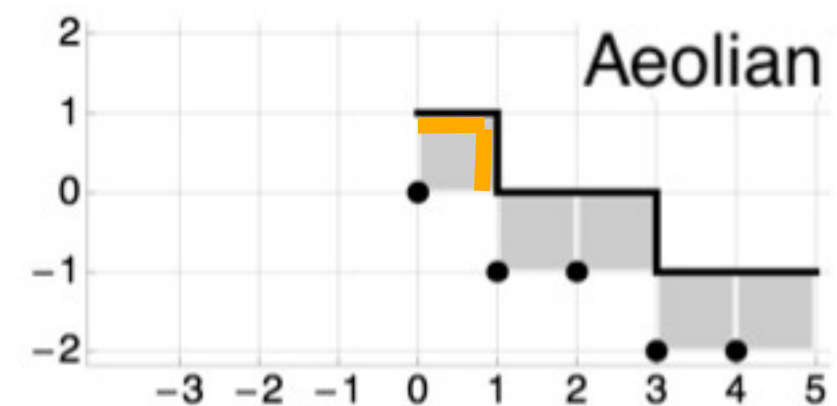
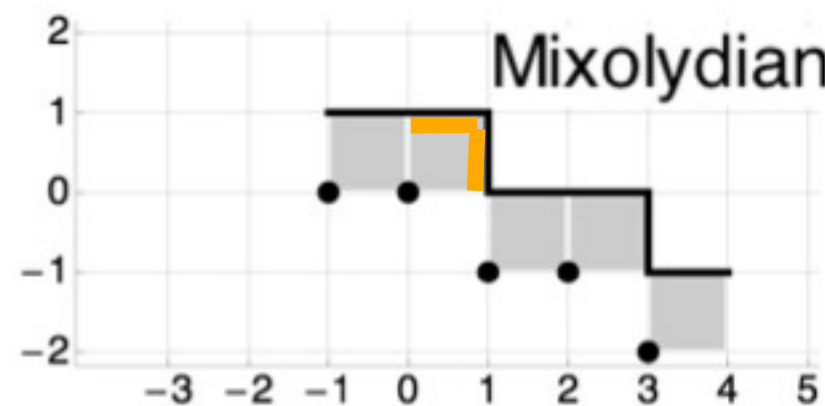
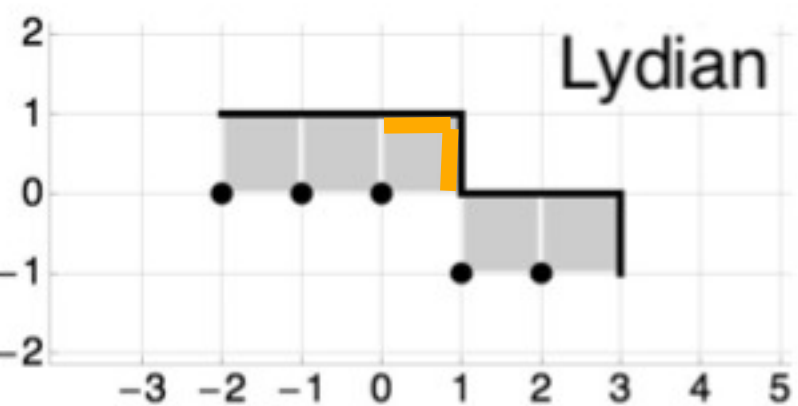
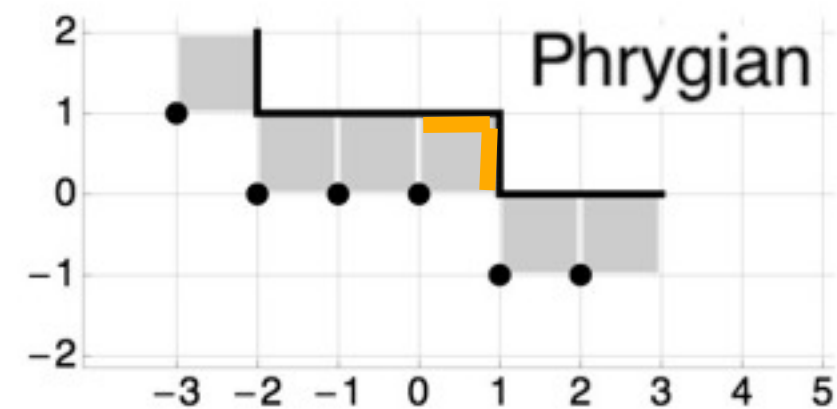
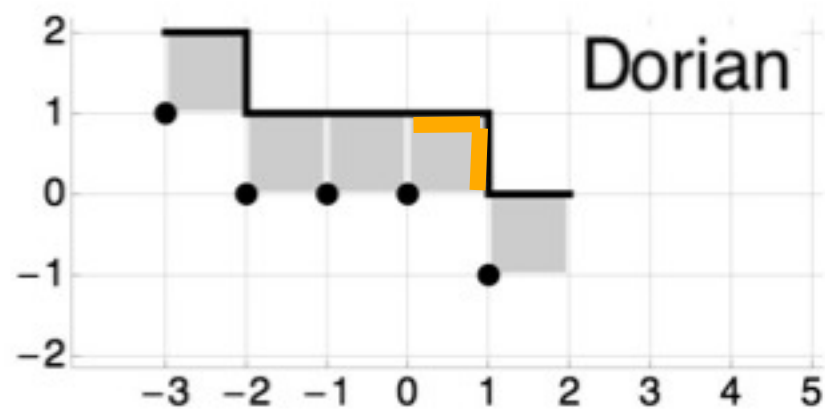
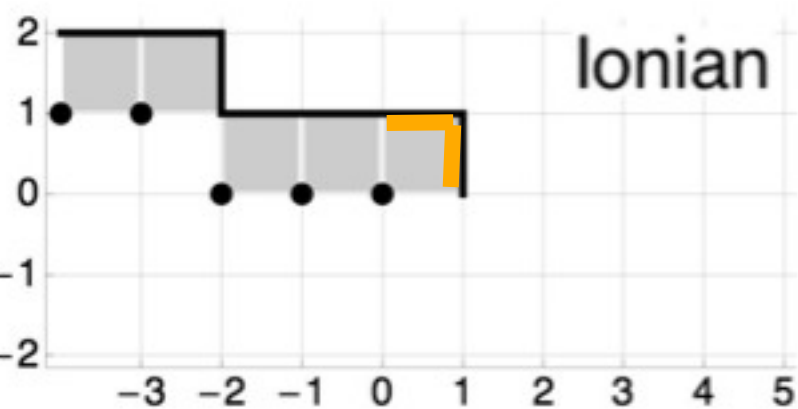




$$E(f_i^*)(\text{Musical Notation})$$

$$\phi$$

$$E(f_i^*)^*(\text{Diagram})$$



Zarlino 1571

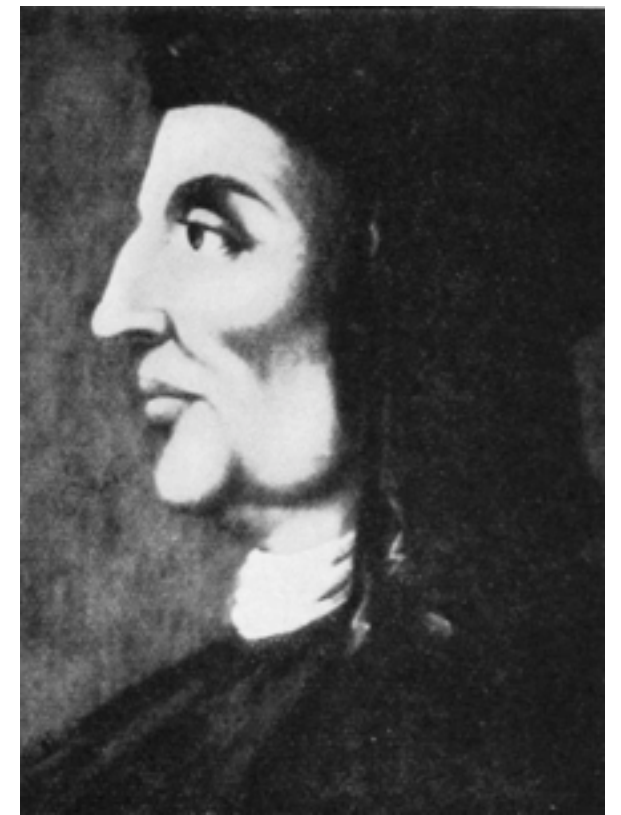
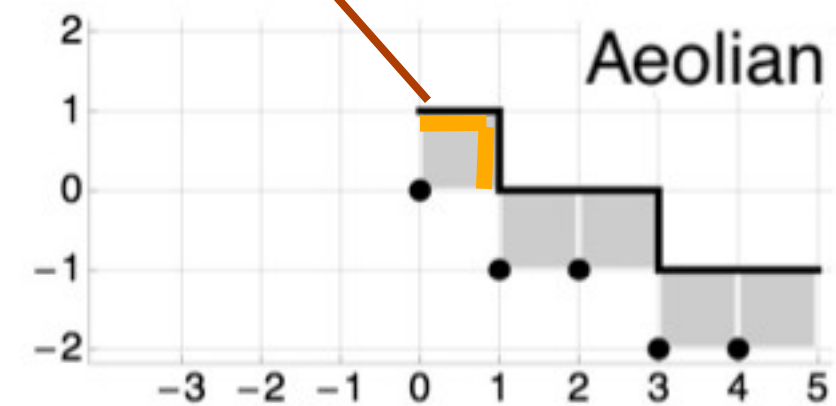
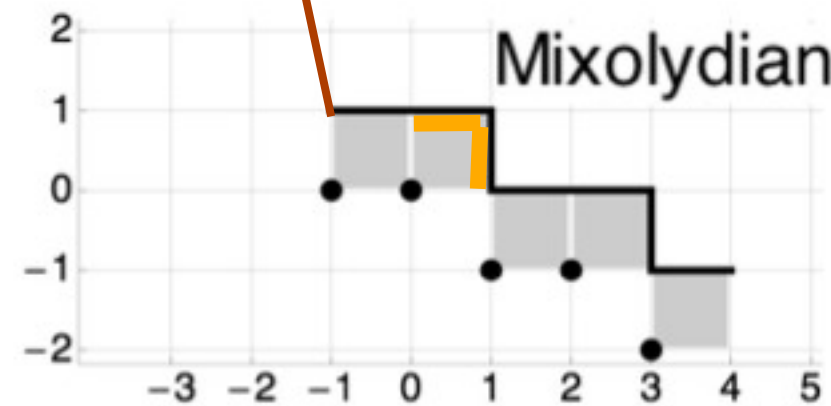
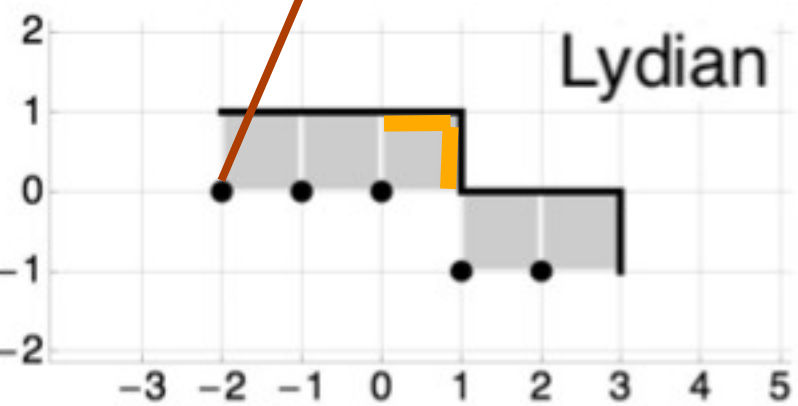
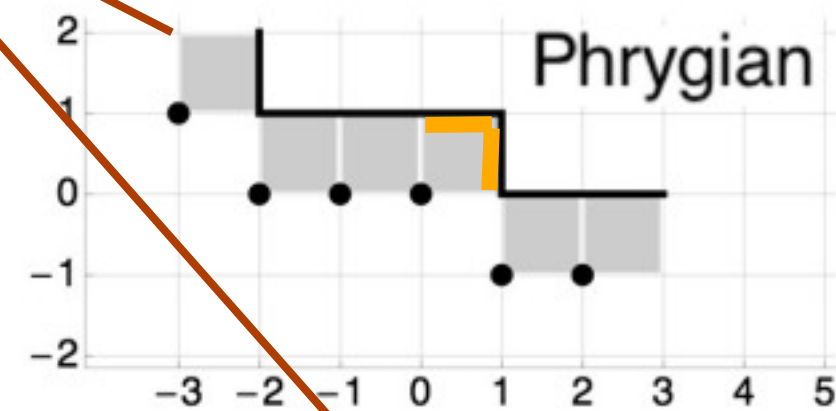
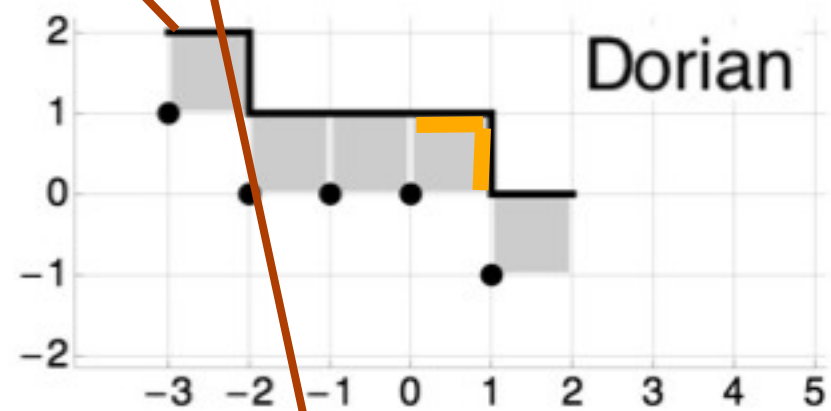
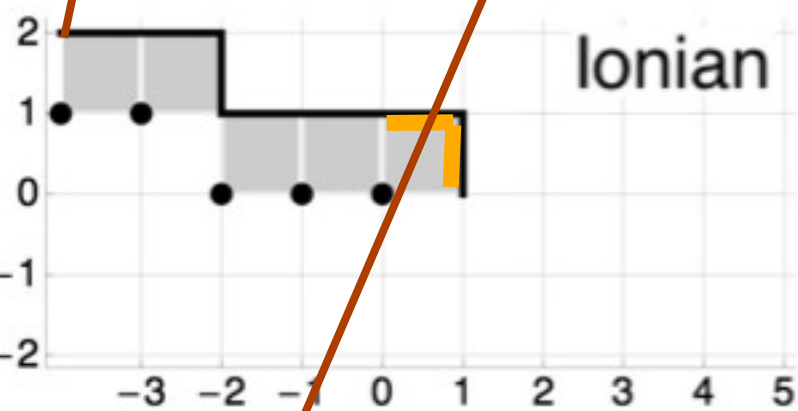
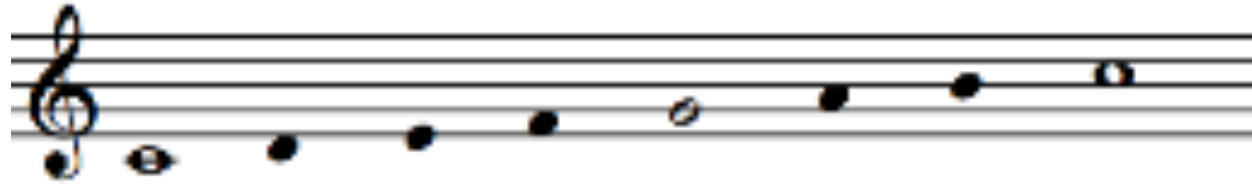


Diagram illustrating the relationship between Zarlino's musical notation and modern frequency ratios. The top staff shows the notes Ut, Re, Mi, Fa, Sol, La. The bottom staff shows the corresponding frequency ratios: y , x , y , x , y . The ratios are grouped into two sets: p (a, a, b, a) and q (a, a), and p' (y, x, y, x, y) and q' (y, x, y, x, y). The ratios are connected to the notes by lines: Ut to y , Re to x , Mi to y , Fa to x , Sol to y , and La to y .

$E(f_i^*)^*($



Ionian Substitution

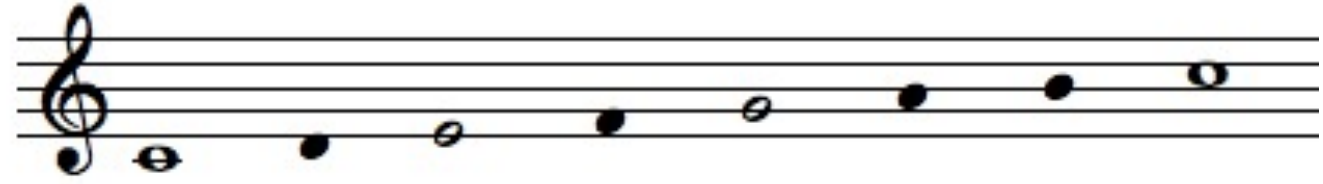


$$\sigma : \{a, b\}^* \rightarrow \{a, b\}^*$$

$$\begin{aligned} a &\mapsto aabb, \\ b &\mapsto aab. \end{aligned}$$

$$E_0(\sigma) = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

Major Substitution

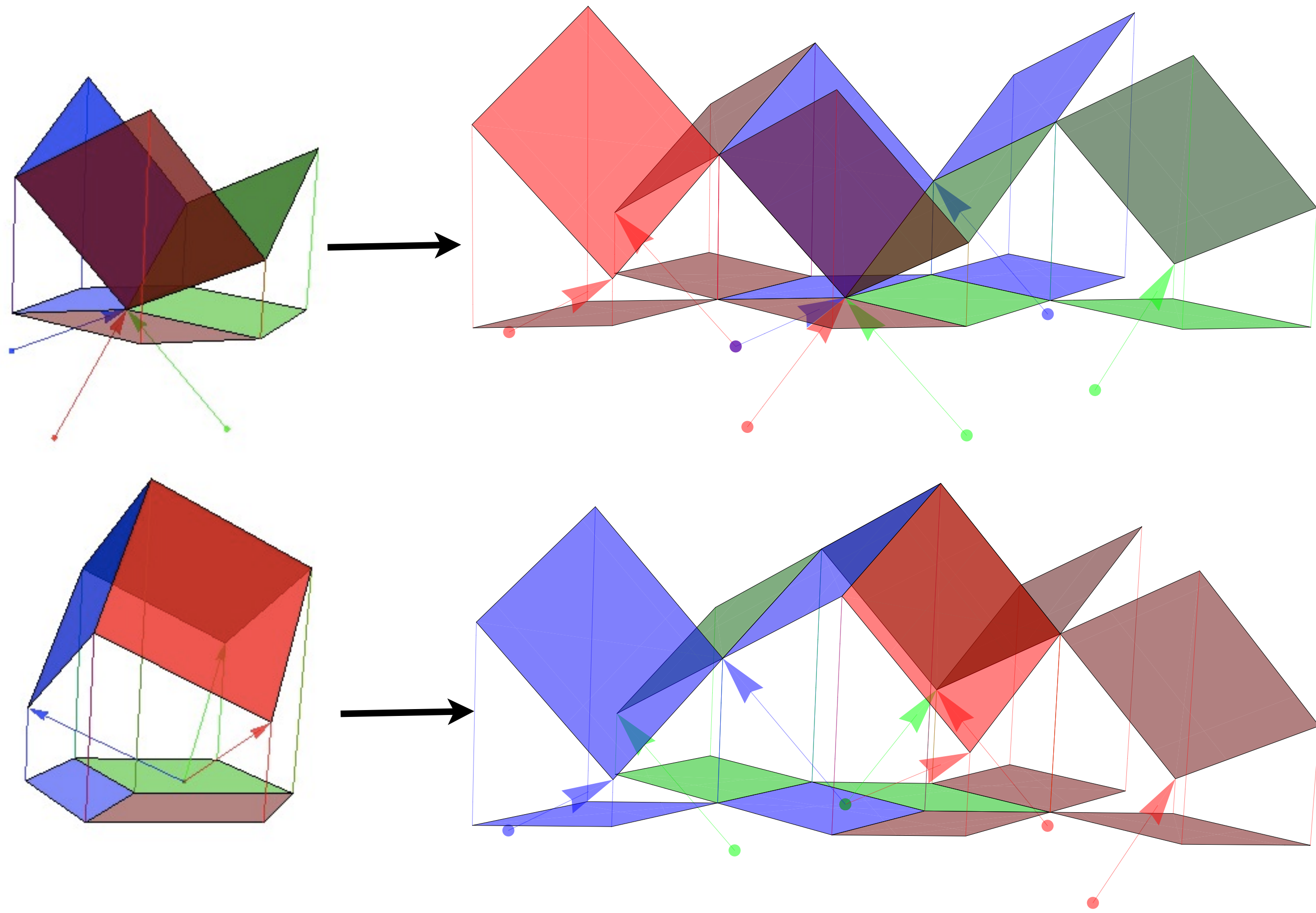


$$\sigma : \{a, b, c\}^* \rightarrow \{a, b, c\}^*$$

$$\begin{aligned} a &\mapsto ab, \\ b &\mapsto ca, \\ c &\mapsto bac. \end{aligned}$$

$$E_0(\sigma) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Dual Map



Dynamical System

