Some stories about small octagons

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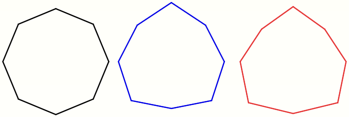
ENSEEIHT-LAPLACE, Toulouse, France

November 2013



RAIM 2013 - IHP - Paris

Presentation Outline

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Outline

Some Definitions

Definition

A $n-$gon is a polygon with n sides and n vertices.

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The diameter of a $n-$gon is longest distance between two vertices.

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A small $n-$gon is a $n-$gon with a diameter 1.

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A_n denote the area of a small $n-$gon and P_n its perimeter.

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$A_n^{=}$ denote the area of an equilateral small $n-$gon and $P_n^{=}$ its perimeter.

Definition (Four Problems)

- Which small polygons have the maximal area?
- Which small polygons have the maximal perimeter?
- Which equilateral small polygons have the maximal area?
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Reinhardt's results 1922

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The regular $n-$gons have all the properties of maximal perimeter and area, for n odd.

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For all n, a bound for the perimeter is

$$P_n \leq 2n \sin \frac{\pi}{2n}$$

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For all n, a bound for the area is

$$A_n \leq \frac{1}{2} \times n \times \left(\frac{1}{2\cos\frac{\pi}{2n}}\right)^2 \times \sin\frac{2\pi}{n}$$

The bounds are reached when n is *odd*.

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Small Quadrilateral Polygons: Maximal Area

- Maximal area $n = 4$:

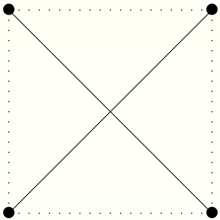

Figure : $A_4^{=} = A_4 = \frac{1}{2}$.

With $\frac{b \times h_1}{2} + \frac{b \times h_2}{2} = \frac{b \times (h_1 + h_2)}{2} = \frac{1}{2}$, $b = 1$ and $h_1 + h_2 = 1$.

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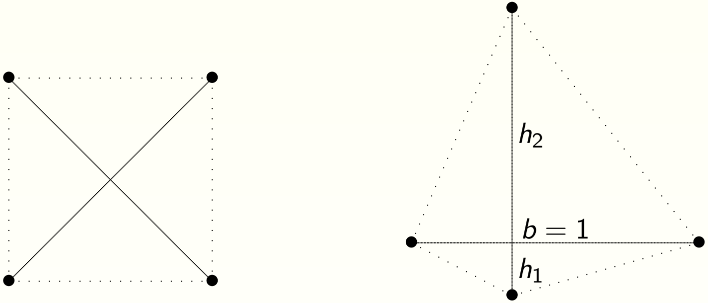

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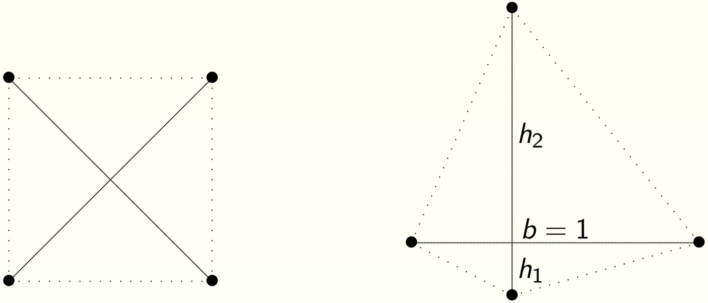

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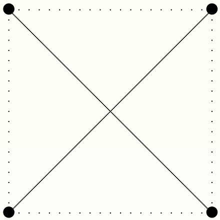

$P_4^{=} = 2\sqrt{2} \approx 2.8284$

$P_4 = 2 + 4\sin\frac{\pi}{12} \approx 3.0353$

Result from Tamvakis 1987 (and Datta 1997).

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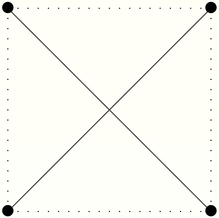

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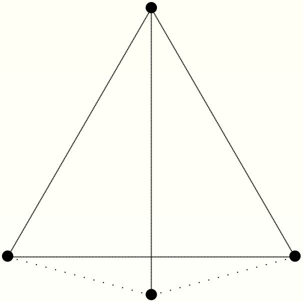

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Problem of P_∞ and Reuleaux Polygons

$$P_\infty := \begin{array}{ll} \max & \text{Perimeter} \\ \text{s.t.} & \text{Diameter} = 1 \\ & \text{The set is convex} \end{array}$$

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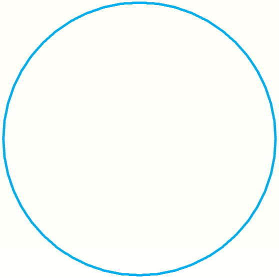

Disk

$P_\infty = \pi$

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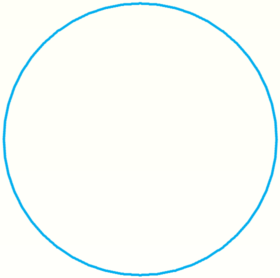

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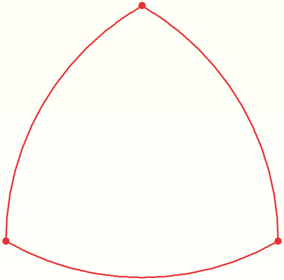

Reuleaux triangle

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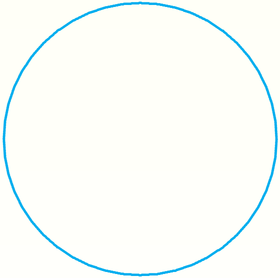

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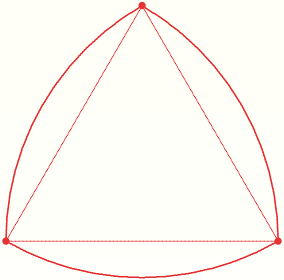
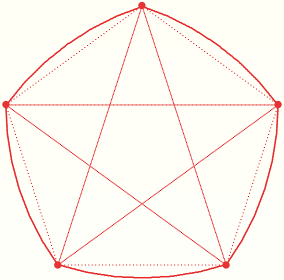


Reuleaux triangle

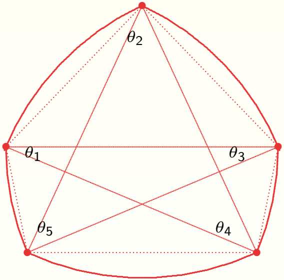
$P_\infty = 3(\frac{\pi}{3}) = \pi$

Odd Reuleaux polygons : figures of constant width



Reuleaux regular pentagon

$P_\infty = 5(\frac{\pi}{5}) = \pi$



Reuleaux irregular pentagon

$P_\infty = \sum_{i=1}^{5} \theta_i = \pi$

Reuleaux Polygons Main Property and Applications

Same width in every directions:

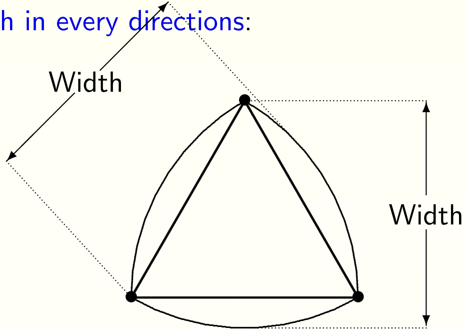


Figure : Example of width of a Reuleaux triangle.

Applications:

- A Reuleaux triangle is used in the SMART car (for the injection system)!
- For the design of a dollar: 1$ Canadian is a Reuleaux polygon with eleven sides.
  See "A $1 Problem" paper in AMM of Mossinghoff.

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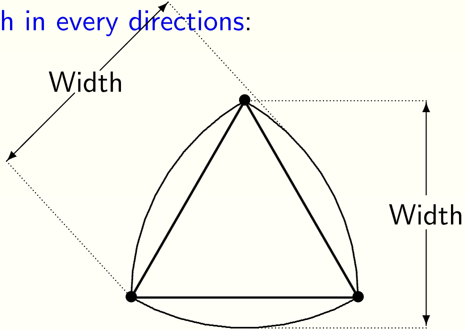

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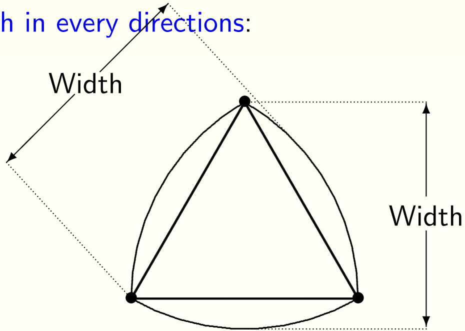

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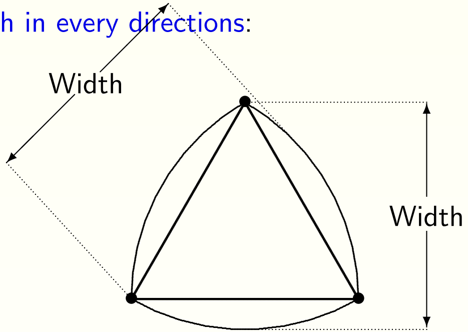

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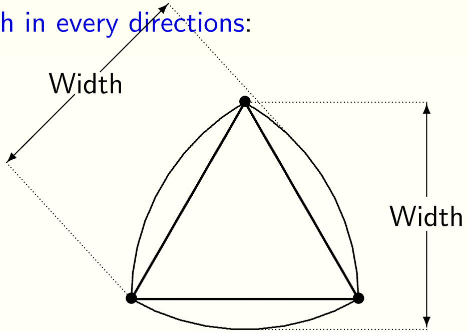

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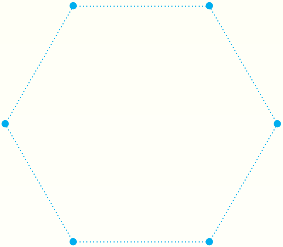
Examples of coins

Hexagon with Maximal Perimeter

The upper bound of $2n\sin(\frac{\pi}{2n})$ is attained for irregular n-gons.



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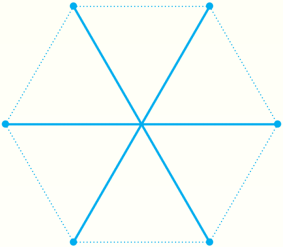

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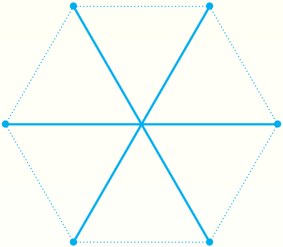

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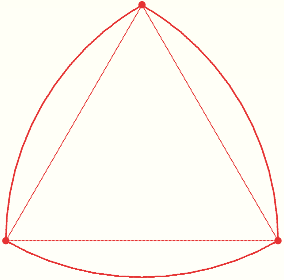

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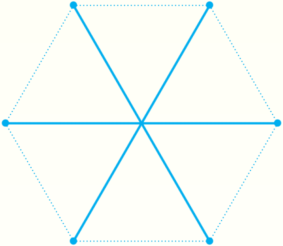

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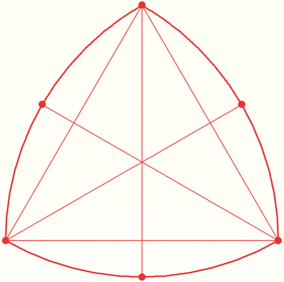

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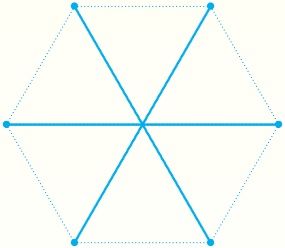

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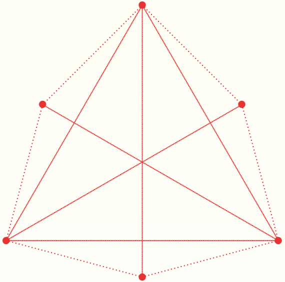

Optimal hexagon

$P_6 = P_6^{=} = 12sin(\frac{\pi}{12}) \approx 3.10582854$

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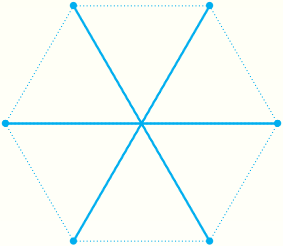

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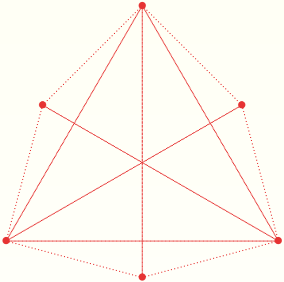

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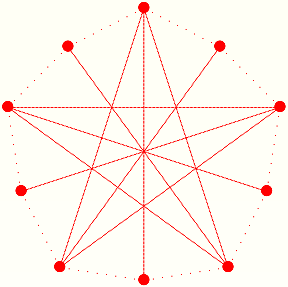
When n is not a power of 2,

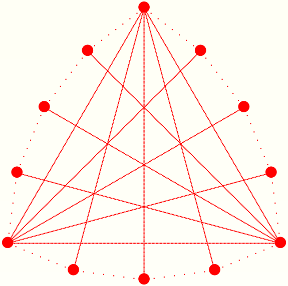
$2n\sin(\frac{\pi}{2n}) \leq \max P_n^{=} \leq \max P_n \leq 2n\sin(\frac{\pi}{2n})$.

This result is due to Vincze 1952.

Examples of Maximal Perimeter Solutions when n is not a Power of 2



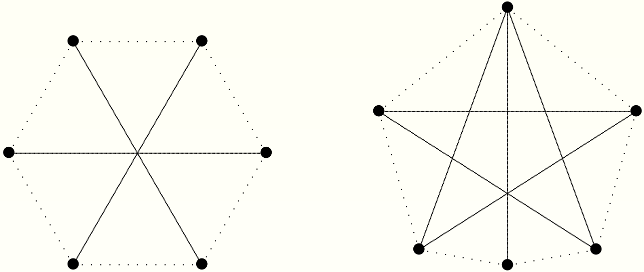
Optimal Decagon
$P_{10} = P_{10}^{=} \approx 3.1287$



Optimal Dodecagon
$P_{12} = P_{12}^{=} \approx 3.1326$

Figure : Examples of polygons with maximal perimeter when n is even but $n \neq 2^s$.

Graham's Hexagon with Maximal Area, 1975



Regular Hexagon

$A_6^{=} = \frac{3\sqrt{3}}{8} \approx 0.6495$

Graham's Hexagon

$A_6 \approx 0.6750$

Figure : Two hexagons with maximal area.

Gain about 3.9% (comparing to the regular hexagon).

Diameter Graph and Geometric Reasoning

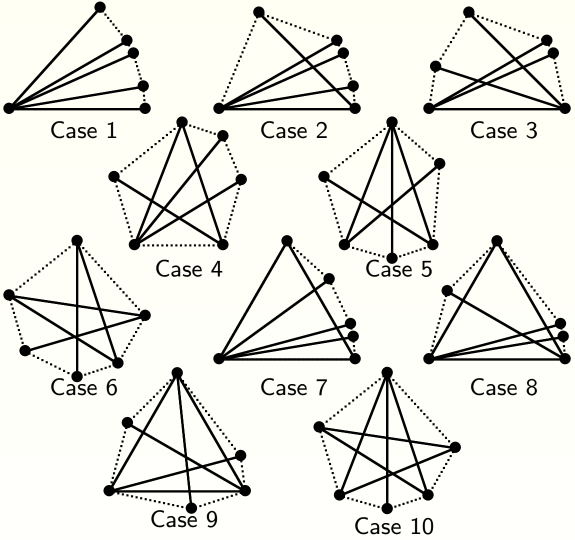


Figure : The ten possible diameter configurations for the hexagon

Case 10 with Symmetry Hypothesis

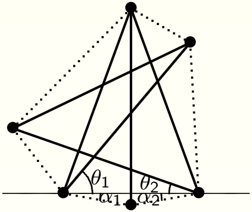


Figure : Configuration 10 of area R_1.

Hypothesis of symmetry ? Graham wrote in his paper:

> *"It is immediate that in order to maximize area R_1, it is necessary that $\alpha_1 = \alpha_2$. It is slightly less immediate (but equally true) that it is also necessary that $\theta_1 = \theta_2$. (The details are not particularly interesting and are omitted)."*

$\Longrightarrow$ solve a global optimization problem in one variable.

Graham's, Bieri's or Yuan's Hexagon ?

Retrospectively with hypothesis of symmetry, Bieri answers to this question in 1961. 14 years before Graham!

Title: "Ungelöste Probleme: Zweiter Nachtrag zu Nr. 12" (Open Problem, second supplement to number 12) answering to Lenz: "Ungelöste Probleme Nr. 12" posed in 1956 in *Elemente der Mathematik*.

This remark come from Mossinghoff: "a 1$ problem", AMM.

Bao Yuan give a complete proof in his Report of Master Degree in 2004: "The Largest Small Hexagon".

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Solved problems and open problems

- The solutions to P_∞ are the disk and all odd Reuleaux polygons.
- When n is odd, the regular n-gon solves the four problems of maximal perimeter and area.
- When $n = k2^s$ for k odd and s integer, then the k-gon with extra vertices solves the both problems for the perimeter.
- When $n = 4$, the square solves $P_4^{=} = 2\sqrt{2} \approx 2.828427$ and the following solves $P_4 = 2 + 4\sin(\frac{\pi}{12}) \approx 3.035276$.
- When $n = 6$, the Graham's hexagon solved max A_6 and the regular hexagon solve max $A_6^{=}$; Vincze's hexagon (based on a Reulaux triangle) solved max P_6 and max $P_6^{=}$.

Open cases, when $n = 8$: the Octagons.

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- When $n = 4$, the square solves $P_4^{=} = 2\sqrt{2} \approx 2.828427$ and the following solves $P_4 = 2 + 4\sin(\frac{\pi}{12}) \approx 3.035276$.
- When $n = 6$, the Graham's hexagon solved $\max A_6$ and the regular hexagon solve $\max A_6^{=}$; Vincze's hexagon (based on a Reulaux triangle) solved $\max P_6$ and $\max P_6^{=}$.

Open cases, when $n = 8$: the Octagons.

Optimal Small Octagons

- For the area and the perimeter $\Longrightarrow$ 31 diameter graphs.
- About 10 cases can be discarded by geometric reasoning.
- Algorithms: Branch and Cut algorithm for quadratic non-convex programs for the area -CPU−times: about 100h- and IBBA for the perimeter -56h (44h for discarding case 18)-

Solutions:

Optimal Octagon (31)
$A_8 \approx 0.726867$

Optimal Octagon (29)
$P_8 \approx 3.121147\ldots$

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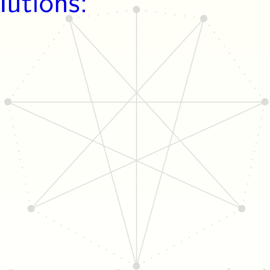

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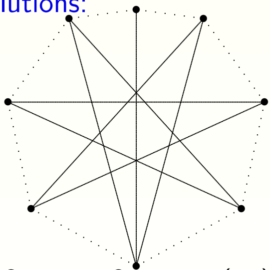

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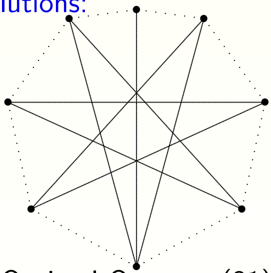

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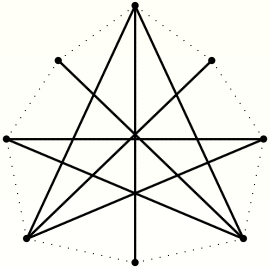

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Optimal Equilateral Small Octagons (perimeter)

- For the area, the regular octagon is optimal.
- For the perimeter, Vincze published in 1952 a better one: (due to his wife)

Vcinze's Wife's Octagon

$(P_8^=) \approx 3.0912...$

Optimal Octagon

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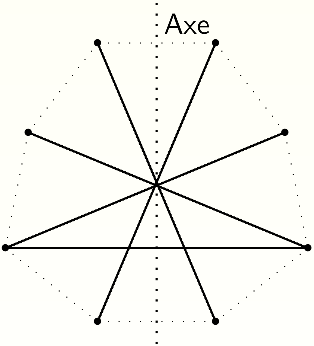

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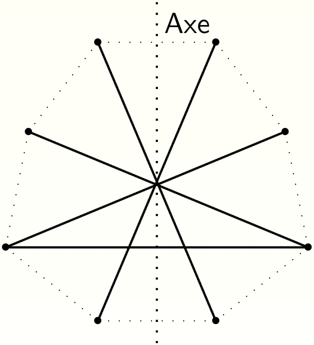

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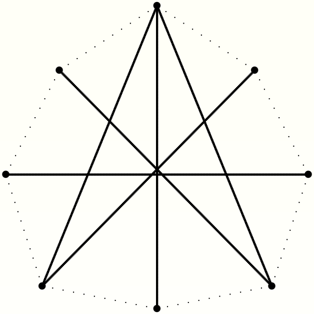

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Presentation Outline

Problem Formulation for the Largest Small Octagon

Formulation by a nonconvex quadratic program:

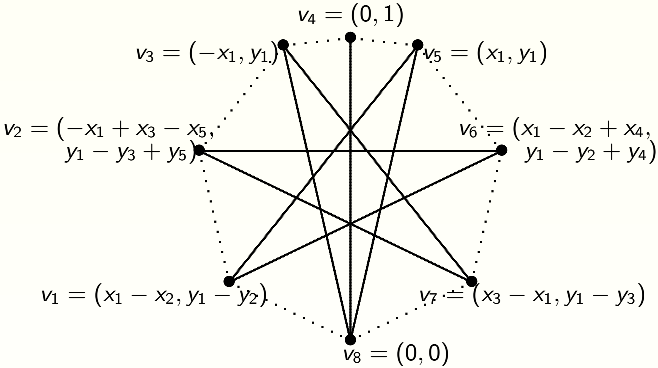

Figure : Case of $n = 8$ vertices. Definition of variables.

A nonconvex quadratic program

$$
\left\{
\begin{array}{ll}
\max\limits_{x,y} & \frac{1}{2}\{(x_2+x_3-4x_1)y_1+(3x_1-2x_3+x_5)y_2 \\
& +(3x_1-2x_2+x_4)y_3+(x_3-2x_1)y_4 \\
& +(x_2-2x_1)y_5\}+x_1 \\
& \\
\text{s.t.} & (2x_1-x_2-x_3+x_4+x_5)^2+(y_2-y_3+y_4-y_5)^2=1 \\
& (x_3-2x_1+x_2)^2+(x_7-x_8)^2\leq 1 \\
& x_i^2+y_i^2=1, i=1,2,3,4,5 \\
& x_2-x_3\geq 0 \\
& y\geq 0 \\
& 0\leq x_1\leq 0.5 \\
& 0\leq x_i\leq 1, \quad i=2,3,4,5.
\end{array}
\right.
$$

Numerical solutions

Soft.	Year	Accuracy	CPU-time
QP	1997	10^{-4}	100h
Gloptipoly	2010	$10^{-7}*$	5s
IBBA	2013	$10^{-8}*$	171s

- QP : $A_8^* \approx 0.726867$
- Gloptipoly: $A_8^* \in [0.72686845, 0.72686849]$
- IBBA: $A_8^* \in [0.72686848797 32928, 0.7268684897329281]$

Solution: $x_1 = 0.26214172, x2 = 0.67123417, x3 = 0.67123381, x4 = 0.90909242, x5 = 0.90909213$

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Solvers

- QP : (RLT) $x_i x_j \to w_{ij}$ + McCormick constraints ($w_{ij} \leq .\& \geq .$).
- Gloptipoly:
  SDP relaxation (find polynomial bases - 2nd relaxation)
  VSDP - rigorous upper bounds.
- IBBA:

$$\left\{ \begin{array}{ll} \max\limits_{x \in X \subseteq \mathbb{R}^n} & f(x) \\ \text{s.t.} & g_i(x) \leq 0, \\ & h_j(x) = 0, \end{array} \right. \to \left\{ \begin{array}{ll} \max\limits_{x \in X^{\mathcal{F}} \subseteq \mathcal{F}^n} & f^{\mathcal{F}}(x) \\ \text{s.t.} & g_i^{\mathcal{F}}(x) \leq \epsilon_g^{\mathcal{F}}, \\ & h_j^{\mathcal{F}}(x) \in [-\epsilon_f^{\mathcal{F}}, \epsilon_f^{\mathcal{F}}], \end{array} \right.$$

$$(P) \leq (P_R)$$

Lower bounds : $QP = 0.726867$, $Gloptipoly = 0.72686845$, $IBBA = 0.726868479732928$

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About bounds ?

$$\text{UB (decreases by its)} \geq (P_R) \geq (P) \geq LB?$$

LB:

$A_8^{\mathcal{F}}$: floating point solution of (P_R).

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$A_8^{\mathcal{F}} \simeq (P)$?

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Lower bound: a formulation

Remark: The solution is almost symmetric!

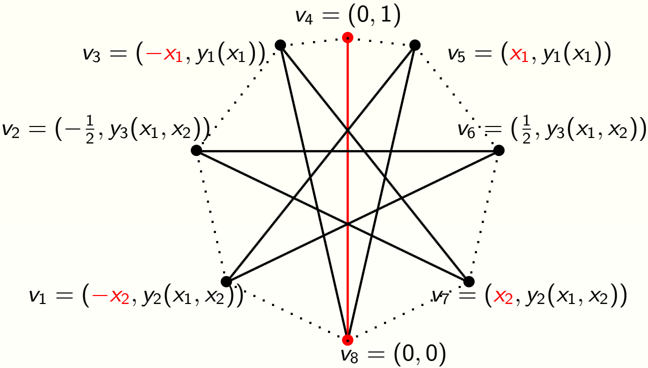

Figure : Symmetric case. Definition of variables.

A nonconvex program for LB

$$\begin{cases} \max\limits_{x_1,x_2} & x_2 y_3 - \frac{1}{2} y_2 + \frac{1}{2} y_1 - x_1 y_3 + x_1 \\ & \\ & 0 \leq x_1 \leq 0.5 \\ & 0 \leq x_2 \leq 0.5. \end{cases}$$

Where

$$y_1(x_1) = \sqrt{1 - x_1^2}$$
$$y_2(x_1, x_2) = y_1(x_1) - \sqrt{1 - (x_1 + x_2)^2}$$
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IBBA $\longrightarrow A_8^S \in 0.7268684827516[265, 365]$

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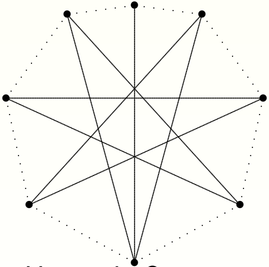
Conclusion on the Hansen's Octagon

- 31 diameter graphs: Graham's conjecture is proved $\longrightarrow$ 1 case (31). Foster and Szabo Results (2007).
- Bounds:

$$A_8^* \in 0.72686848275[16265, 26265]$$

$$Gloptipoly = 0.726868\mathbf{4}9, \; IBBA = 0.7268684897329$$

Solutions:



Hansen's Octagon

Area ≈ 0.72686848275

Application of the Hansen's Octagon