Interval Analysis: Nonlinear Computer-Assisted Proofs

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Introduction

Numerical analysis

- Solve numerical problems (systems of equations, optimization problems, dynamical systems simulations, etc.)
- Using computers (simple operations computed with finite precision)

⇒ Approximate solutions, or worth artefact solutions

⇒ Central question: Sensitivity analysis

Interval analysis

- Compute rigorously using finite precision computations (interval arithmetic)

→ Powerful when small scale, strongly nonlinear (some applications in robotics, control, etc)

→ Rigorous proofs of mathematical statements (up to algorithmic errors, or compilers errors, or hardware errors)

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Famous Computer Assisted Proofs Involving IA

- **Thomas Hales**: Kepler conjecture (1998)
- **Warwick Tucker**: The Lorentz attractor exists, answering the 14^{th} Smale problem (1999)

Session Content

- **Alexandre Goldsztejn**: "La fée clochette est chaotique" (2011)
- **Nicolas Delanoue**: "Classification des applications lisses d'un domaine simplement connexe de $\mathbb{R}^2$ dans $\mathbb{R}^2$" (2013)
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Outline

1 Interval Analysis
- Rigorous Approximate Computations
- Computing Rigorously With Sets
- Proving Existence of Solutions

2 Tinkerbell Is Chaotic

Interval Arithmetic: Rigorous Approximate Computations

Approximate Computations

- $\frac{1}{3} \Rightarrow$ 0.33333333333334 (enough in most situations)
- Rump's function:

$$(333.75-a^2)b^6+a^2(11a^2b^2-121b^4-2)+5.5b^8+\frac{a}{2b}$$

evaluate with $a = 77617$ and $b = 33096$ using double precision

$\Rightarrow$ 1.172 (or 0 on my computer!)

- True result: -0.827396

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Interval Arithmetic: Rigorous Approximate Computations

Interval Arithmetic

- $[\underline{x},\overline{x}] \circ [\underline{y},\overline{y}] = [\underline{z},\overline{z}]$ defined to contain all possible results
- Example: $[0,1]+[3,4]=[3,5]$, $[0,1]/[3,4]=[0,0.33333333333334]$
- General explicit formula:

$$[\underline{x},\overline{x}]+[\underline{y},\overline{y}] = [\downarrow \underline{x}+\underline{y} \downarrow, \uparrow \overline{x}+\overline{y} \uparrow]$$

$$\begin{aligned}[\underline{x},\overline{x}]\times[\underline{y},\overline{y}] = [\ & \min\{\downarrow \underline{x}\underline{y}\downarrow, \downarrow \overline{x}\underline{y}\downarrow, \downarrow \underline{x}\overline{y}\downarrow, \downarrow \overline{x}\overline{y}\downarrow\}, \\ & \max\{\uparrow \underline{x}\underline{y}\uparrow, \uparrow \overline{x}\underline{y}\uparrow, \uparrow \underline{x}\overline{y}\uparrow, \uparrow \overline{x}\overline{y}\uparrow\}\ ]\end{aligned}$$

  Explicit formulae for $\frac{x}{y}$, $\exp x$, $\log x$, $\cos x$, etc.

Enclosure of Rounding Errors

- $\frac{[1,1]}{[3,3]} = [\frac{1}{3},\frac{1}{3}] \ \Rightarrow\ [0.33333333333333, 0.33333333333334]$
- Rump's function: $[-3.89\times10^{22}, 3.66\times10^{22}]$

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   - Proving Existence of Solutions
     - Mono-Variable Interval Newton
     - Intermediate Value Theorem
     - Poincaré-Miranda Theorem
2. Tinkerbell Is Chaotic

Interval Arithmetic: Computing Rigorously With Sets

Interval Evaluation of an Expression

- $f(x, y) = xy + 3x$ can be evaluated with interval arguments
  e.g. for $x = [1, 2]$ and $y = [-1, 1]$:
  $$f([1, 2], [-1, 1]) = [1, 2] \times [-1, 1] + 3 \times [1, 2] = [1, 8]$$
- **Fundamental theorem of interval analysis**: The interval evaluation contains all possible results
  $$[1, 8] \supseteq \{xy + x : x \in [1, 2], y \in [-1, 1]\}$$

! Pessimism $\Rightarrow$ Curse of dimensionality

Direct Application

- Proved: $f(x, y) \geq 0$ for all $x \in [1, 2]$ and $y \in [-1, 1]$

$\Rightarrow$ Checking function sign over domains (robust stability, etc.)

- Proved: $f(x, y) \neq 0$ for all $x \in [1, 2]$ and $y \in [-1, 1]$

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Mono-Variable Interval Newton

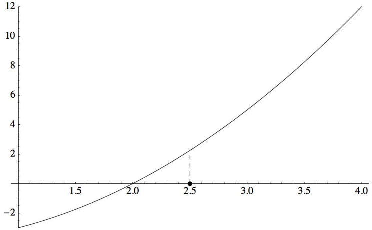
The Interval Newton Method

- Given an initial interval
- All possible derivatives in the interval ⇒ ”cone”

→ Computed using an interval evaluation of the derivative

→ Mean-value theorem ⇒ function's graph included inside cone

- Solution ∈ intersection of cone and x-axis
- *Old interval strictly contain new interval ⇒ existence proof!*



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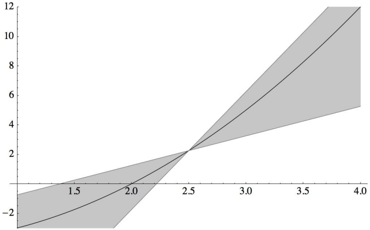

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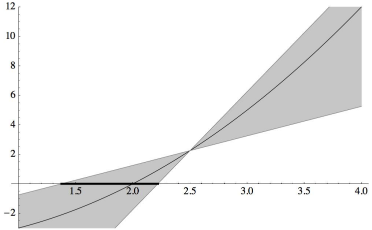

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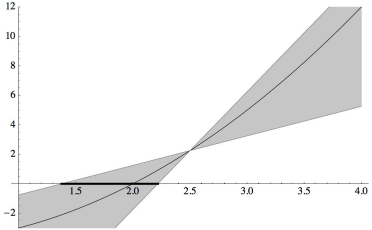

Intermediate Value Theorem

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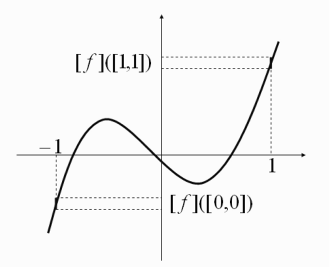
$$f(-1) \leq 0 \wedge f(1) \geq 0 \implies (\exists x \in [-1,1])(f(x) = 0)$$



Usage with Interval Extensions

- $[f]([-1,-1]) \leq 0 \Rightarrow f(-1) \leq 0$
- $[f]([1,1]) \geq 0 \Rightarrow f(1) \geq 0$

$\rightarrow$ $(\exists x \in [-1,1])(f(x) = 0)$

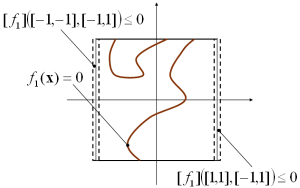
Poincaré-Miranda Theorem

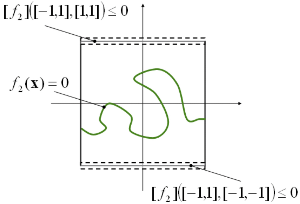
Poincaré-Miranda Theorem (Early 20th Century)

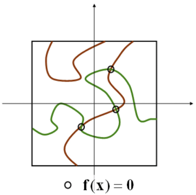
- Poincaré-Miranda theorem ($\approx$ Brouwer fixed point theorem)
- Check signs of the function taken inside the sides of the boxes

$$\begin{pmatrix} \forall i \in \{1,\dots,n\} \\ \forall x \in [-1,1]^n \end{pmatrix} \begin{pmatrix} x_i = -1 \Rightarrow f_i(x) \leq 0 \\ x_i = 1 \Rightarrow f_i(x) \geq 0 \end{pmatrix}$$

$$\Longrightarrow\ (\exists x \in [-1,1]^n)(f(x) = 0)$$







Outline

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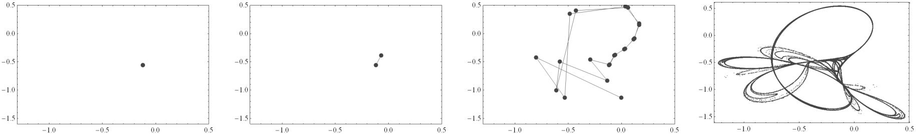
2 Tinkerbell Is Chaotic

Continuous State Discrete Time Dynamical System

Definitions

- Dynamical system: $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$
- Initial value: $y_0 \in \mathbb{R}^n$
- Orbit: $(y_0, y_1, \ldots)$ or $(\ldots, y_{-1}, y_0, y_1, \ldots)$ with $y_{k+1} := f(y_k)$



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Prove rigorously that the system is chaotic
$\rightarrow$ Revisit Stoffer and Palmer 1999 technique
(which requires injectivity, but fairies are not injective in general)

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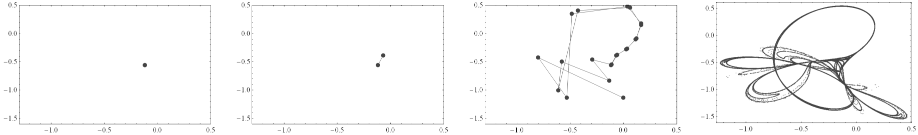

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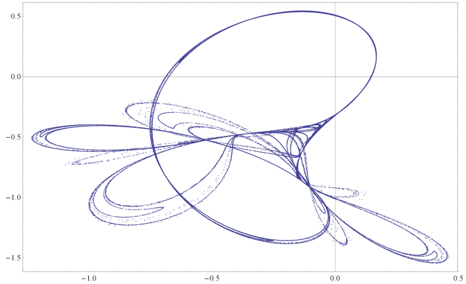
Paradoxical Plotting of Strange Attractors

Problem

- We think we plotted Tinkerbell's strange attractor $\rightarrow$ self contradiction!
- The plot uses double precision and several hundred thousands of steps
- Strange attractor $\Rightarrow$ chaos $\Rightarrow$ exponential divergence $\Rightarrow$ plot completely false!



Shadowing

Pseudo Orbits

- δ-pseudo orbit: $(\ldots, y_{-1}, y_0, y_1, y_2, \ldots)$ such that $||y_{k+1} - f(y_k)|| \leq \delta$

  $\delta \equiv$ computation precision (`double` with $y_k \approx 1 \Rightarrow \delta \approx 10^{-16}$)

Forward Error Analysis

- Forward error: $||y_k - f^k(y_0)||$
- Chaotic system $\Rightarrow ||y_k - f^k(y_0)||$ grows exponentially

$\rightarrow$ Forward error analysis useless

Shadowing: A Backward Error Analysis

- Used in hyperbolic systems theory
- Simulation accurate for a slightly perturbed initial condition
- Tinkerbell's strange attractor has a shadow

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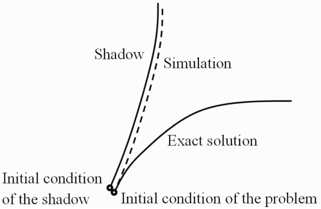

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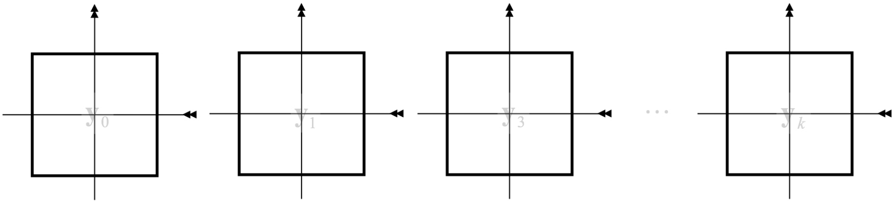
Finite Length Shadows

The Inductive Containment Property

- Pseudo orbit + small surrounding boxes
- Hyperbolicity: *n* linearly independant directions that are either contracting or expanding
- Checked using interval evaluations: Inductive Containment Property

⇒ Exists an orbit inside the boxes (Poincaré-Miranda Theorem)



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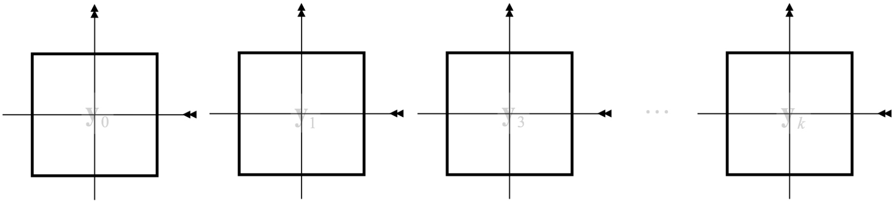

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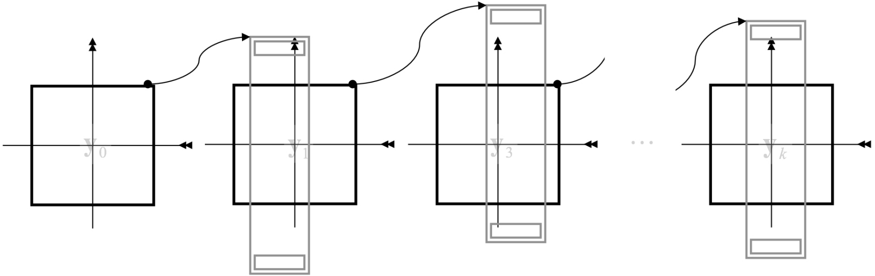

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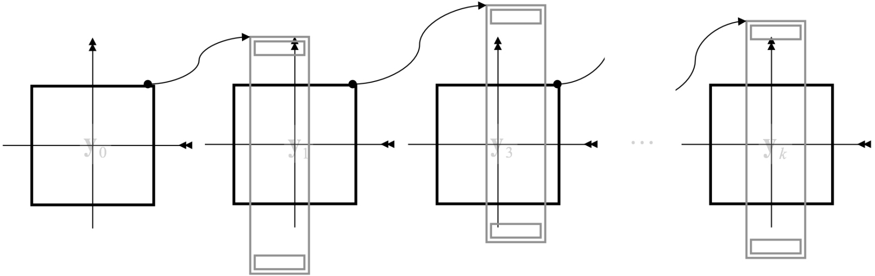
- Hyperbolicity: *n* linearly independant directions that are either contracting or expanding
- Checked using interval evaluations: Inductive Containment Property

$\Rightarrow$ Exists an orbit inside the boxes (Poincaré-Miranda Theorem)



Infinite Length Shadows: Periodic Pseudo-Orbit

Infinite Length Shadow

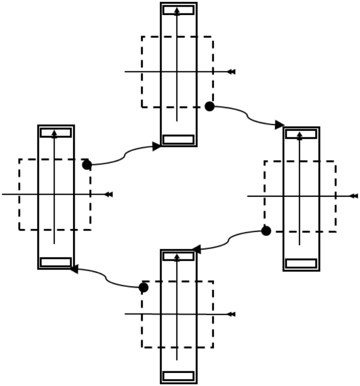
- δ-pseudo periodic orbit
- We prove ICP rigorously for $(x_0, x_1, \ldots, x_m, x_0)$

$\Rightarrow$ Infinite length δ-pseudo orbit

$$(\ldots, x_0, x_1, \ldots, x_m, x_0, x_1, \ldots)$$

also verifies the ICP

$\Rightarrow$ Infinite length shadow using finite number of interval computations



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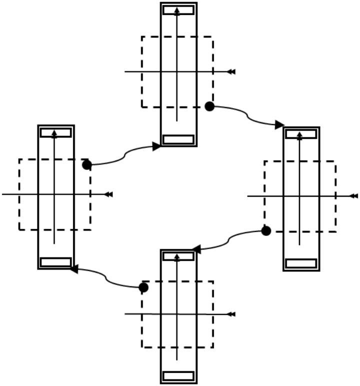

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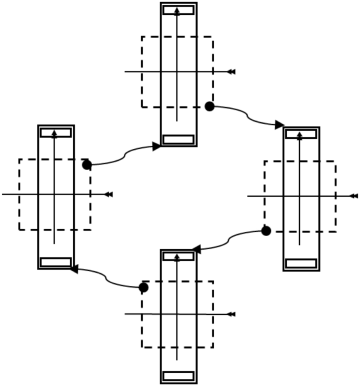


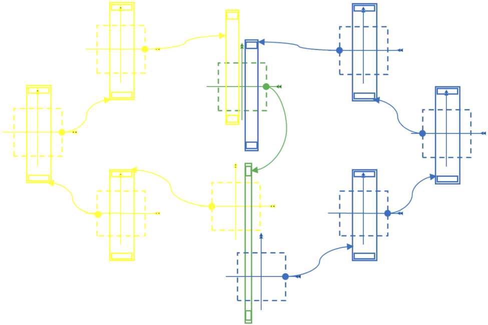
Infinite Length Shadows: Branching Pseudo-Orbit

Following Stoffer and Palmer 1999

- Two δ-pseudo periodic orbit: $||x_0 - f(x_m)|| \leq \delta$ and $||y_0 - f(y_m)|| \leq \delta$ such that $x_0 \approx y_0$
- We prove ICP for $(x_0, x_1, \dots, x_m, x_0)$ and $(y_0, y_1, \dots, y_m, y_0)$

  the same box is used for x_0 and y_0

$\Rightarrow$ Also valid for all infinite length pseudo orbits



Chaos

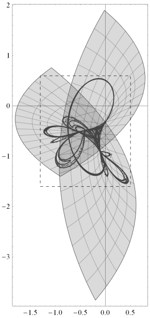
- Each $\cdots 01001 \cdots$ defines an orbit $(z_k)_{k \in \mathbb{Z}}$:
  - $z_0 \in [x_0]$
  - Very $\neq$ to each others
  - $\rightarrow$ Very high sensitivity to initial conditions, while bounded

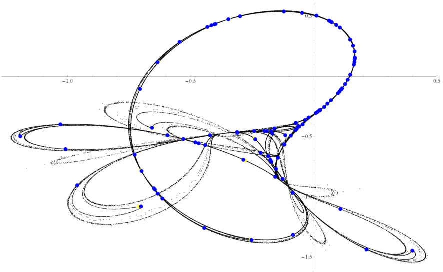
$\Rightarrow$ Formal proof of chaos using symbolic dynamics

Tinkerbell Is Chaotic

Tinkerbell Map

- $f(x) = (0.9x_1 + x_1^2 - 0.6013x_2 - x_2^2, 2x_1 + 0.5x_2 + 2x_1x_2)$ (not injective)
- We have found two pseudo periodic orbits of period 37, which satisfy:
  - $||x_0 - y_0|| \approx 2.5 \times 10^{-9}$ and $\max_i\{||x_i - y_i||\} \approx 9.1 \times 10^{-3}$
  - Inductive containment property





Tinkerbell is chaotic !