

Improving Modular Inversion in RNS using the Plus-Minus Method

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Context and Objectives

Research group main objective:

Design hardware implementations of cryptoprocessor for ECC (elliptic curve cryptography) on FPGA and ASIC

Various aspects of arithmetic operators for ECC:

- algorithms
- **representations** of numbers
- hardware implementations

This work

Modular inversion operators in the **residue number system** (RNS)

My Ph. D. objectives:

- RNS natural parallelism $\rightarrow$ speed
- RNS natural support for randomization $\rightarrow$ protection against some side-channel attacks (SCA)

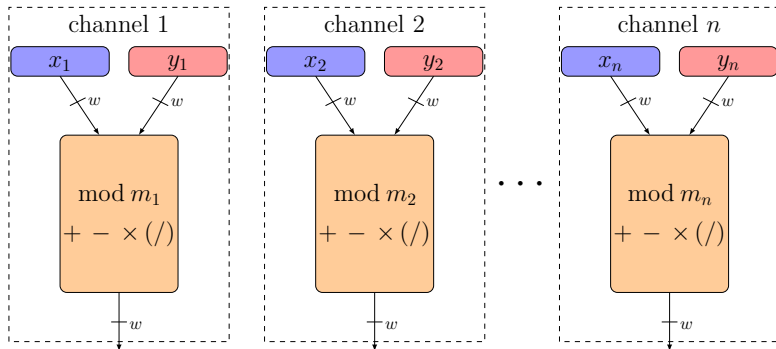
Residue Number System (RNS) [9] [4]

X and Y two $\mathbb{F}_P$ elements (160–600 bits) are represented by:

$$\vec{X} = (x_1, \dots, x_n) = (X \bmod m_1, \dots, X \bmod m_n)$$

$$\vec{Y} = (y_1, \dots, y_n) = (Y \bmod m_1, \dots, Y \bmod m_n)$$

Modular operations over w -bit chunks, e.g. w is 16–64



RNS base $\mathcal{B} = (m_1, \dots, m_n)$, n pairwise co-prime integers of w bits
with $n \times w \geq \log_2 P$

Pros:

- **Carry-free** between channels
 - each channel is independant
- **Fast parallel** $+$, $-$, $\times$ and some exact divisions
 - computations over all channels can be performed in parallel
 - a multiplication requires n modular multiplications of w -bit words
- **Non-positional** number system
 - randomization of computations (SCA countermeasures)

Cons:

- comparison, **modular reduction** and division are **hard**

Base Extension [10]

- Usual technique for modular reduction: add redundancy using **2 bases**
- $\mathcal{B} = (m_1, \dots, m_n)$ and $\mathcal{B}' = (m'_1, \dots, m'_n)$ are coprime RNS bases
- X is $\vec{X}$ in $\mathcal{B}$ and $\vec{X}'$ in $\mathcal{B}'$
- The **base extension** (BE , introduced in [10]) is defined by:

$$\vec{X}' = BE(\vec{X}, \mathcal{B}, \mathcal{B}')$$

- Some operations become possible after a base extension
 - $M = \prod_{i=1}^n m_i$ is **invertible** in $\mathcal{B}'$
 - **exact division by M** can be done easily
- State-of-art BE algorithms cost $n^2 + n$ w -bit modular multiplications

RNS Montgomery Reduction (RNS-MR) [8][1]

Input: $\vec{X}$, $\vec{X}'$ with $X < \alpha P^2 < PM$ and $2P < M'$

Output: $(\vec{\omega}, \vec{\omega}')$ with $\omega \equiv X \times M^{-1} \bmod P$
 $0 \leq \omega < 2P$

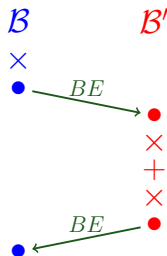
$$\vec{Q} \leftarrow \vec{X} \times (-\vec{P}^{-1}) \quad (\text{in base } \mathcal{B})$$

$$\vec{Q}' \leftarrow BE(\vec{Q}, \mathcal{B}, \mathcal{B}')$$

$$\vec{S}' \leftarrow \vec{X}' + \vec{Q}' \times \vec{P}' \quad (\text{in base } \mathcal{B}')$$

$$\vec{\omega}' \leftarrow \vec{S}' \times \vec{M}^{-1} \quad (\text{in base } \mathcal{B}')$$

$$\vec{\omega} \leftarrow BE(\vec{\omega}', \mathcal{B}', \mathcal{B})$$



RNSMR cost: $2n^2 + O(n)$ w-bit modular multiplications

How to exploit RNS properties?

Maximize the use of fully parallelizable operations, e.g. computing patterns in the form of $(AB + CD) \bmod P$

State-of-Art RNS Inversion Algorithm

State-of-art RNS modular inversion (FLT-RNS) [5, 3]:

- based on **Fermat's Little Theorem** (FLT): $X^{-1} = X^{P-2} \bmod P$
- requires a large exponentiation
- involves **a lot of modular reductions**
- parallelization is **very limited** due to data dependencies

FLT-RNS complexity: $O(\log_2 P \times n^2)$ multiplications of w -bit words

Binary Extended Euclidean from [7]§ 4.5.2

Input: $X, P \in \mathbb{N}$, $P > 2$ with $\gcd(X, P) = 1$

Output: $|X^{-1}|_P$

$(U_1, U_3) \leftarrow (0, P)$, $(V_1, V_3) \leftarrow (1, X)$

while $V_3 \neq 1$ *and* $U_3 \neq 1$ **do**

while $|V_3|_2 = 0$ **do**

$V_3 \leftarrow \frac{V_3}{2}$

if $|V_1|_2 = 0$ **then** $V_1 \leftarrow \frac{V_1}{2}$ **else** $V_1 \leftarrow \frac{V_1 + P}{2}$

while $|U_3|_2 = 0$ **do**

$U_3 \leftarrow \frac{U_3}{2}$

if $|U_1|_2 = 0$ **then** $U_1 \leftarrow \frac{U_1}{2}$ **else** $U_1 \leftarrow \frac{U_1 + P}{2}$

if $V_3 \geq U_3$ **then** $V_3 \leftarrow V_3 - U_3$, $V_1 \leftarrow V_1 - U_1$

else $U_3 \leftarrow U_3 - V_3$, $U_1 \leftarrow U_1 - V_1$

if $V_3 = 1$ **then return** $|V_1|_P$ **else return** $|U_1|_P$

Extended Euclidean algorithms are not used due to comparison and division costs in RNS

Proposed Algorithm (PM-RNS) (1/4)

Our proposition is based on binary extended Euclidean algorithm, where comparisons are **replaced by cheaper operations** using Plus-Minus (PM) trick [2]:

- if X and Y are odd then $X + Y = 0 \bmod 4$ or $X - Y = 0 \bmod 4$

We only choose **odd** moduli:

- multiplication by $4^{-1} \leftrightarrow$ division by 4

To use Plus-Minus trick we must define a **cheap mod4**

Our proposition must be efficient on **the state-of-art architecture** of ECC with RNS, **reuses and adapts existing blocks**

Proposed Algorithm (PM-RNS) (2/4)

Input: $\vec{X}, P > 2$ with $\gcd(X, P) = 1$

Output: $\vec{S} = |\vec{X}^{-1}|_P, S < 2P$

Initialisations

while $\widehat{V}_3 \neq \pm 1$ and $\widehat{U}_3 \neq \pm 1$ **do**

while $|b_{V_3}|_2 = 0$ **do**

if $b_{V_3} = 0$ **then** $r \leftarrow 2$ **else** $r \leftarrow 1$

$\widehat{V}_3 \leftarrow \text{div2r}(\widehat{V}_3, r), \widehat{V}_1 \leftarrow \text{div2r}(\widehat{V}_1, r)$

$b_{V_3} \leftarrow \text{mod4}(\widehat{V}_3), b_{V_1} \leftarrow \text{mod4}(\widehat{V}_1), v \leftarrow v + r$

$\widehat{V}_3^* \leftarrow \widehat{V}_3, \widehat{V}_1^* \leftarrow \widehat{V}_1$

if $|b_{V_3} + b_{U_3}|_4 = 0$ **then**

$\widehat{V}_3 \leftarrow \text{div2r}(\widehat{V}_3 + \widehat{U}_3, 2), \widehat{V}_1 \leftarrow \text{div2r}(\widehat{V}_1 + \widehat{U}_1, 2)$

$b_{V_3} \leftarrow \text{mod4}(\widehat{V}_3), b_{V_1} \leftarrow \text{mod4}(\widehat{V}_1)$

else

$\widehat{V}_3 \leftarrow \text{div2r}(\widehat{V}_3 - \widehat{U}_3, 2), \widehat{V}_1 \leftarrow \text{div2r}(\widehat{V}_1 - \widehat{U}_1, 2)$

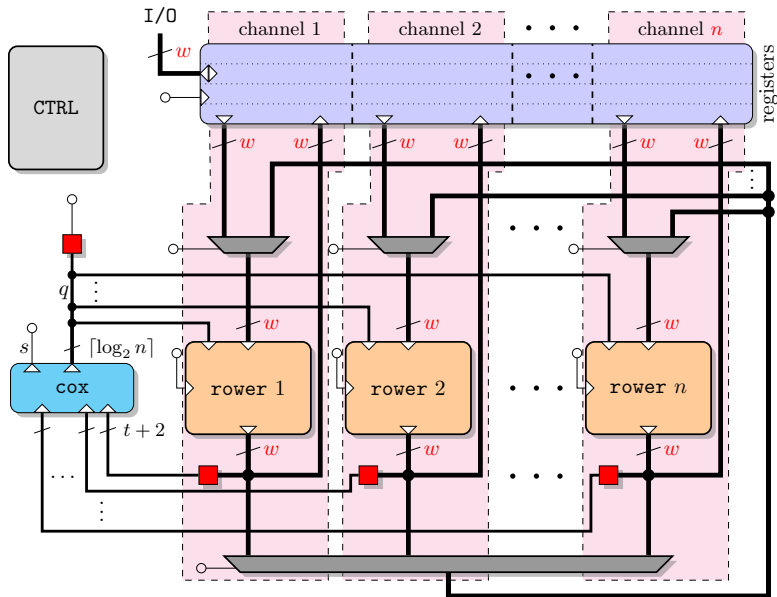
$b_{V_3} \leftarrow \text{mod4}(\widehat{V}_3), b_{V_1} \leftarrow \text{mod4}(\widehat{V}_1)$

if $v > u$ **then** $\widehat{U}_3 \leftarrow \widehat{V}_3^*, \widehat{U}_1 \leftarrow \widehat{V}_1^*, u \leftrightarrow v$

$v \leftarrow v + 1$

Final corrections

Global Architecture (adaptation of [5])



Proposed Algorithm (PM-RNS) (3/4)

To use Plus-Minus trick we must define a **cheap mod4** using CRT:

$$|X|_4 = \left| \left| \sum_{i=1}^n |x_i \cdot M_i^{-1}|_{m_i} \cdot M_i \right|_M \right|_4 = |s - q \cdot M|_4 = s - |q \cdot M|_4$$

with $M = \prod_{i=1}^n m_i$, $M_i = \frac{M}{m_i}$ and $q = \left\lfloor \frac{\sum_{i=1}^n |x_i \cdot M_i^{-1}|_{m_i} \cdot M_i}{M} \right\rfloor$

Kawamura's approximation [6] of q :

$$q = \left\lfloor \sum_{i=1}^n \frac{|x_i \cdot M_i^{-1}|_{m_i}}{m_i} \right\rfloor \approx \left\lfloor \sum_{i=1}^n \frac{\text{trunc}(|x_i \cdot M_i^{-1}|_{m_i})}{2^w} \right\rfloor$$

The **Cox** module

- **computes q** : sum of n values of $t = 6$ bits (MSBs)
- **computes s** : sum of n 2-bit values modulo 4

The main differences with [5] comes from the Cox:

- In [5], the CRT sum was performed in **n cycles**, here in 1
- In [5], the Cox **only computes q** (not s)

Proposed Algorithm (PM-RNS) (4/4)

Kawamura's approximation of q **requires** $0 \leq X < (1 - err_{max})M$, with $err_{max} \in [0, 1[$ but PM requires subtractions. For instance $X = |U - V|_M = |-1|_M = M - 1$ so $X > (1 - err_{max})M$

$\hat{X}$ is an affine function of $\vec{X}$ defined as

$$\hat{X} = ((x_1 + c_1) \cdot M_1^{-1}, \dots, (x_n + c_n) \cdot M_n^{-1})_{\mathcal{B}}$$

- $\vec{C} = (c_1, \dots, c_n)$ is 0 mod 4 : $\text{mod}4(\hat{X}) = |X|_4$
- $-P < X < P$ in PM-RNS, so if $P \leq C < (1 - err_{max})M - P$ then $0 \leq \hat{X} < (1 - err_{max})M$
- the factor $(M_1^{-1}, \dots, M_n^{-1})$ is included and **done once** for the computation of q and s

Remark: `div2r` is designed to handle properly $\hat{X}$, i.e. it returns the hat representation of the expected output (for instance $\widehat{(\frac{X+P}{4})}$)

Cost of the Proposed Algorithm

Our proposition:

- On average, $0.71 \log_2 P$ main loop iterations
- PM-RNS works without base extension
- Total average complexity: $O(\log_2 P \times n)$

Example: for 192 bits (number of w -bit modular multiplications):

$n \times w$	FLT-RNS	PM-RNS	Gain Factor
12×17	103140	5474	18
9×22	61884	4106	15
7×29	40110	3193	12

Remark: state-of-art FLT-RNS complexity is $O(\log_2 P \times n^2)$ multiplications of w -bit words

Hardware Implementations and Comparisons

PM-RNS and FLT-RNS have been **fully implemented**:

- using the same CAD tools (ISE 12.4)
- on Virtex 5 FPGAs
- with the same synthesis options and efforts
- and have an optimized implementation (for fair comparison)

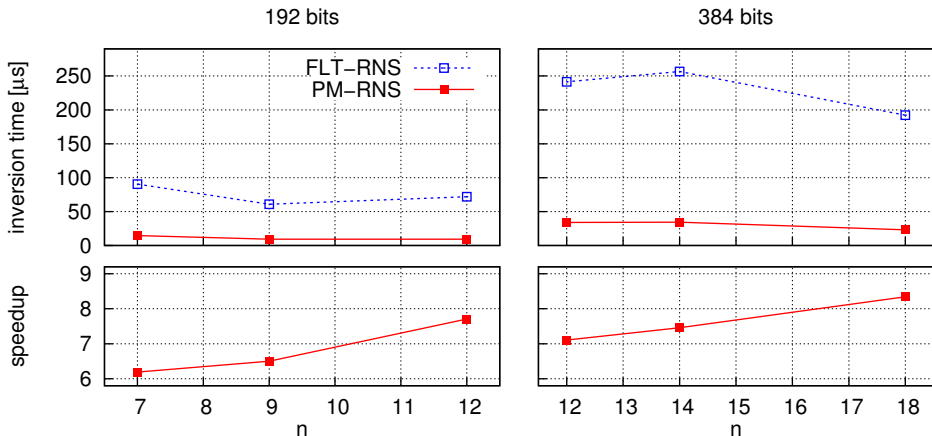
For both algorithms, **2 field sizes** have been implemented:

- 192 bits
- 384 bits

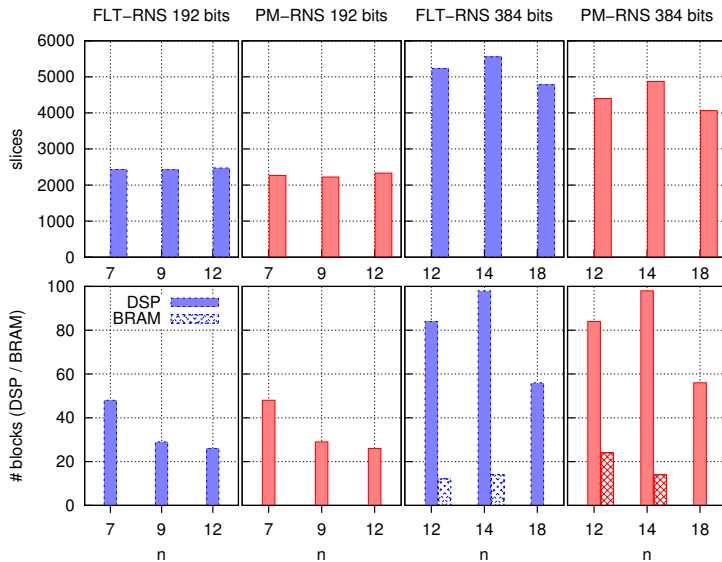
For each size, **3 couples (n, w)** have been implemented

Timing Implementation Comparison

Virtex 5 results with DSP blocks and BRAMs, with $w \in \{17, 22, 29\}$ for 192 bits and $w \in \{22, 29, 33\}$ for 384 bits:



Area Implementation Results Comparison



Conclusion

Our proposition:

- is about 6–8 times **faster** than the best state-of-art solution
- with a **small area overhead** on RNS operator for ECC
- full FPGA optimization and fair comparison

Future works on hardware implementation:

- improve FPGA implementation of the new inversion
- integration in a **complete** ECC processor in RNS
- implementation of **randomization** for a scalar multiplication
- study speed vs. area **trade-offs**

Thank you for your attention

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Costs of RNS Modular Operations

X, Y, P : n words of w bits

Elementary operations: mult. = w -bit multiplication

Operation	RNS (mult.)	Standard (mult.)
XY	$2n$	n^2
$X \bmod P$	$2n^2 + 4n$	$n^2 + n$
$XY \bmod P$	$2n^2 + 6n$	$2n^2 + n$
$(X_1 Y_1 + X_2 Y_2) \bmod P$	$2n^2 + 8n$	$3n^2 + n$
$\left(\sum_{i=1}^k X_i Y_i\right) \bmod P$	$2n^2 + (4 + 2k)n$	$(1 + k)n^2 + n$

Factor 2 in RNS XY is due to the use of 2 bases and BE s

Cost of Inner Loop Operations

For each main loop iteration:

- the inner loop has on average $\frac{2}{3}$ iterations
- $\text{RNS} \times$ and div2r cost n w -bit modular multiplications
- $\text{RNS} +, -$ cost n modular w -bit additions
- mod4 requires n additions of t -bit values and $n + 1$ modulo 4 additions

Note: t is the number of truncated bits in q approximation

Implementation of FLT-RNS

Our FLT-RNS implementation:

- is based on the state-of-art one [5] (with the original Cox)
- uses state-of-art Gandino's algorithm [3] to perform RNS exponentiation
- has a better pipeline filling than state-of-art one [5] (control dedicated to the inversion here)

Operations Count

Algo.	ℓ	$n \times w$	w -bit EMM	w -bit EMA	cox-add	mod4-add
FLT-RNS	192	12×17	103140	85950	6876	0
		9×22	61884	48991	5157	0
		7×29	40110	30083	4011	0
	384	18×22	434322	382617	20682	0
		14×29	273462	233247	16086	0
		12×33	206820	172350	13788	0
FLT-RNS NIST	192	12×17	137520	114600	9168	0
		9×22	85512	65322	6876	0
		7×29	53480	40110	5348	0
	384	18×22	579096	510156	27576	0
		14×29	364616	310996	21448	0
		12×33	275760	229800	18 384	0
PM-RNS	192	12×17	5474	8750	5474	5930
		9×22	4106	6562	4106	4562
		7×29	3193	5104	3193	3650
	384	18×22	16487	26376	16487	17402
		14×29	12823	20514	12823	13738
		12×33	10991	17584	10991	11907

FPGA Implementation Results with Dedicated Hard Blocks

Algo.	ℓ	$n \times w$	Area			Freq. MHz	Number of cycles	Duration μ s
			<i>slices</i> (FF/LUT)	DSP	BRAM			
FLT-RNS	192	12×17	2473 (2995/7393)	26	0	186	13416	72.1
		9×22	2426 (3001/7150)	29	0	185	11272	60.9
		7×29	2430 (3182/6829)	48	0	107	9676	90.4
	384	18×22	4782 (5920/14043)	56	0	178	34359	193.0
		14×29	5554 (5910/16493)	98	14	110	28416	258.3
		12×33	5236 (5710/15418)	84	12	107	25911	242.1
PM-RNS	192	12×17	2332 (3371/6979)	26	0	187	1753	9.3
		9×22	2223 (3217/6706)	29	0	187	1753	9.3
		7×29	2265 (3336/6457)	48	0	120	1753	14.6
	384	18×22	4064 (5932/13600)	56	0	152	3518	23.1
		14×29	4873 (6134/14347)	98	14	102	3518	34.4
		12×33	4400 (5694/12764)	84	24	103	3518	34.1

FPGA Implementation Results without Dedicated Hard Blocks

Algo.	ℓ	$n \times w$	Area			Freq. MHz	Number of cycles	Duration μs
			<i>slices</i> (FF/LUT)	DSP	BRAM			
FLT-RNS	192	12×17	4071 (4043/12864)	4	0	128	13416	104.8
		9×22	4155 (3816/13313)	4	0	122	11272	92.3
		7×29	4575 (3952/15264)	0	0	126	9676	76.7
	384	18×22	7559 (7831/27457)	0	0	163	34359	210.7
		14×29	9393 (7818/30536)	0	0	126	28416	225.5
		12×33	9888 (7640/31599)	0	0	107	25911	242.1
PM-RNS	192	12×17	3899 (4212/12519)	4	0	150	1753	11.6
		9×22	3809 (3986/12782)	4	0	146	1753	12.0
		7×29	4341 (4107/14981)	0	0	141	1753	12.4
	384	18×22	7677 (8053/128306)	0	0	168	3518	20.9
		14×29	9119 (8113/30619)	0	0	127	3518	27.7
		12×33	9780 (7908/31902)	0	0	108	3518	32.5