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A quasi-polynomial algorithm for discrete logarithm in small characteristic

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Discrete logarithm

Definition

Let t and s be two elements in a cyclic group. We call discrete logarithm of s in base t, if it exists, the smallest positive integer x such that

$$t^x = s.$$

Example

DSA signature relies on the difficulty of solving the equation

$$t^x \equiv s \mod p,$$

for a prime p and integers t and s.

Example

Pairing based crypto-systems relies on the difficulty of solving the equation

$$t(X)^x \equiv s(X) \mod \varphi(X),$$

for an irreducible polynomial $\varphi(X)$ in $\mathrm{F}_2[X]$ or $\mathrm{F}_3[X]$.

Motivation

The security of the public key protocols relies on the difficulty of primitives:

1. **factorization (RSA);**
2. **discrete logarithm (DSA);**
3. **elliptic curve discrete logarithm (ECDSA).**
4. ...

$$\begin{tikzcd} & \textbf{factorization} \arrow[r, leftrightarrow, "\textbf{same complexity}"'] & \textbf{discrete log.} \arrow[d, leftrightarrow, "\textbf{is analoguous to}"] \\ \textbf{elliptic discrete log.} \arrow[r, "\textbf{pairing-friendly curves reduce to}"'] & \textbf{discrete log.} & \textbf{in } \mathbb{F}_{2^n} \textbf{ and } \mathbb{F}_{3^n} \end{tikzcd}$$

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- $\log_{t_1} t_2 \cdot \log_{t_2} s = \log_{t_1} s$**; we simply write** $\log s$**;**
- **if** $a \in \mathbb{F}_q^*$**, then** $a^{q-1}=1$**. So** $(q-1)\log a \equiv 0 \mod \ell$**, hence** $\log a \equiv 0 \mod \ell$**.**

Smoothness in $\mathbb{F}_q[x]$

Definition

A polynomial is m-smooth if all its irreducible factors have degree less or equal to m.

Proposition

Put $N_q(n, m)$ the number of degree-n monic m-smooth polynomials.

- $N_q(D, 1)/q^D \approx 1/D!$;
- $N_q(D, \frac{1}{6}D)/q^D = c + o(1)$ for a constant $c > 0$.

idea.

$N_q(D, 1) = \binom{q}{D} + \cdots \approx q^D/D!$. $\square$

Obtaining relations

Example

Take $q = 3$, $k = 5$, $\varphi = x^5 + x^4 + 2x^3 + 1$ and $\ell = 11$ (divisor of $3^5 - 1$). We have

$$\begin{aligned} x^5 &\equiv 2(x+1)(x^3+x^2+2x+1) \mod \varphi \\ x^6 &\equiv 2(x^2+1)(x^2+x+2) \mod \varphi \\ x^7 &\equiv 2(x+2)(x+1)^2 \mod \varphi. \end{aligned}$$

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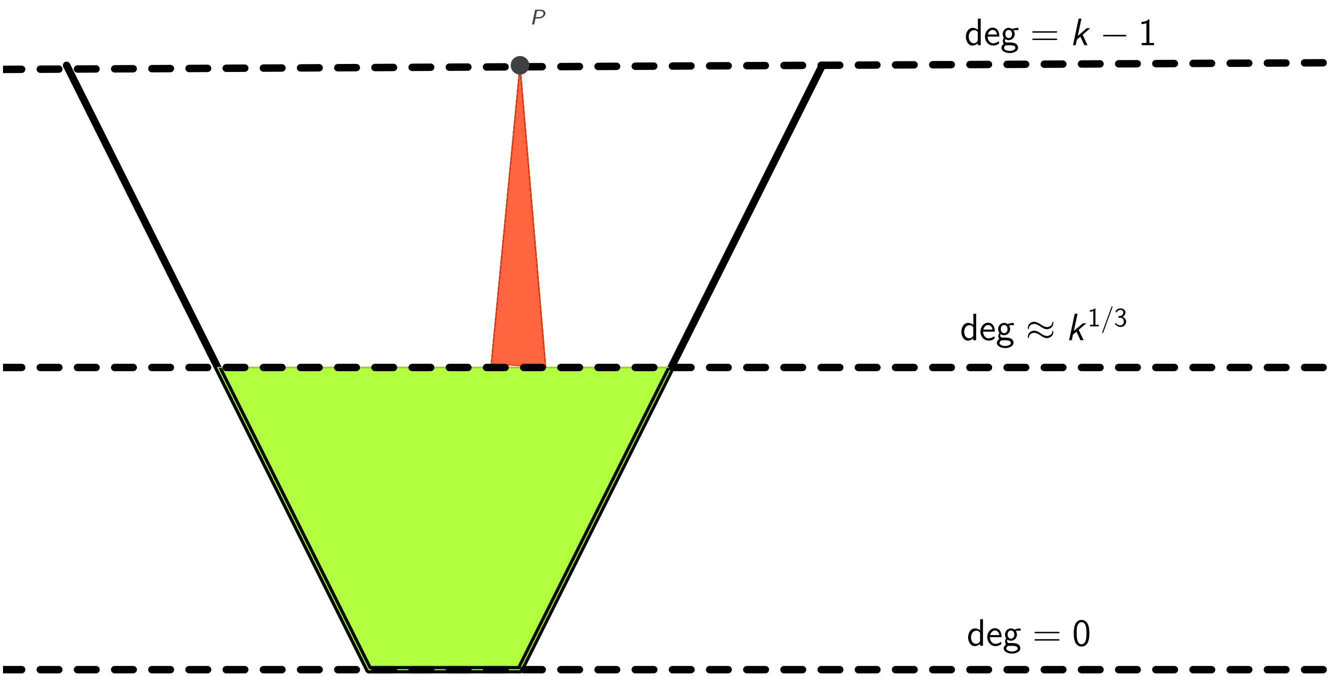
The last relation gives:

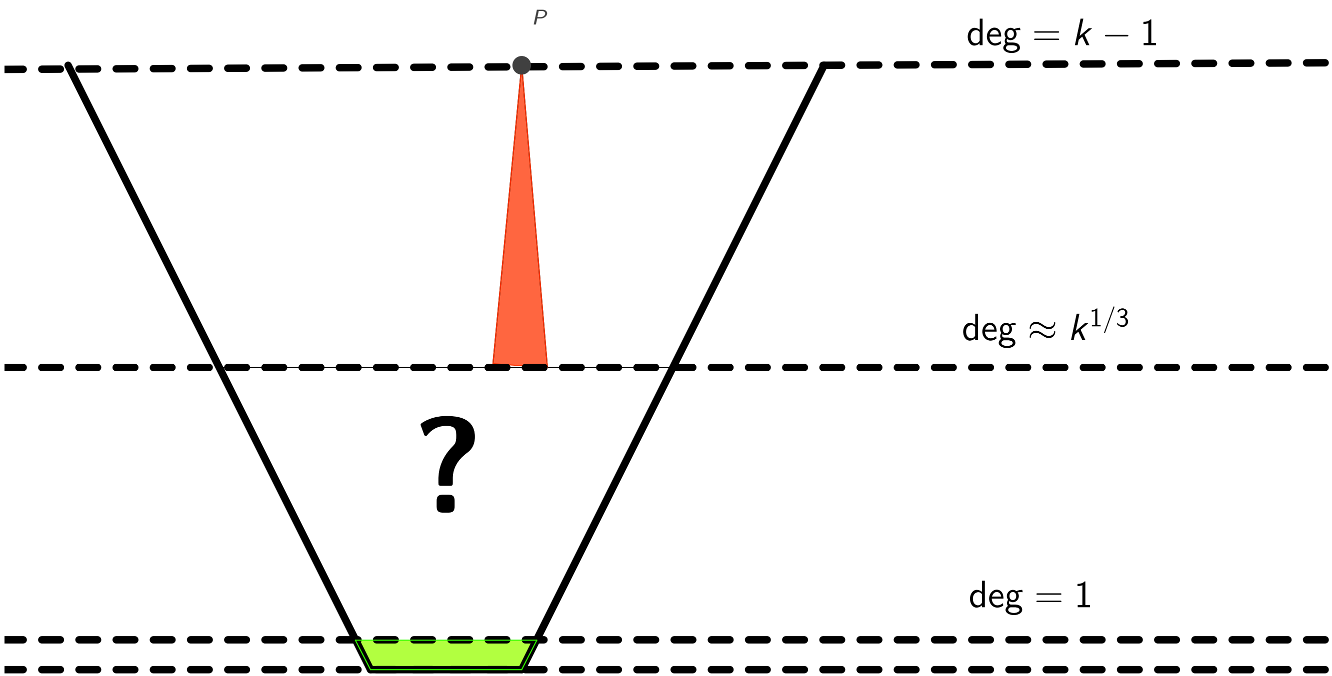
$$7 \log x \equiv 1 \log(x+2) + 2 \log(x+1) \mod 11.$$

With 3 equations we compute $\log x$, $\log(x+1)$ and $\log(x+2)$.

Illustration of the classical algorithms



Speeding up Joux' algorithm



Representing $\mathbb{F}_{q^{2k}}$

Choosing φ: Try random $h_0, h_1 \in \mathbb{F}_{q^2}[x]$ with $\deg h_0, \deg h_1 \leq 2$ until $T(x) := h_1(x)x^q - h_0(x)$ has a divisor of degree k.

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Remark

- The existence of h_0 and h_1 is heuristic but found in practice in time $O(k)$.
- $$h_1(x)x^q \equiv h_0(x) \mod T(x).$$
- If $P \in \mathbb{F}_{q^2}[x]$ then

$$\begin{aligned} h_1(x)^{\deg P} P(x)^q &= h_1(x)^{\deg P} \tilde{P}(x^q) \\ &\equiv h_1(x)^{\deg P} \tilde{P}\left(\frac{h_0}{h_1}\right) \mod T(x). \end{aligned}$$

Building block

Proposition

Under plausible heuristics explained below, for any polynomial P one finds in polynomial time a relation

$$\log P \equiv e_1 \log P_1 + \cdots + e_k \log P_k \mod \ell,$$

with $\deg P_i \leq \frac{1}{2} \deg P$.

Proof: The left hand side

Let $\log P$ be the required computation. For random $a, b, c, d \in \mathbb{F}_{q^2}$ we have

$$h_1(x)^{\deg P}\left((aP+b)^q(cP+d)-(aP+b)(cP+d)^q\right) \equiv \text{ small degree } \mod T(X).$$

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If the small degree polynomial is smooth we obtain

$$\log\left((aP+b)^q(cP+d)-(aP+b)(cP+d)^q\right) \equiv e_1 \log P_1 + \cdots + e_n \log P_n \mod \ell,$$

with $\deg P_i \leq \frac{1}{2}\deg P$.

The right hand side

Recall the identity

$$x^q - x = \prod_{\alpha \in \mathrm{F}_q} (x - \alpha).$$

It gives $x^q y - y^q x = \prod_{(\alpha \in \mathrm{F}_q \bigcup \{\infty\}} (x - \alpha y)$ and

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$$\begin{aligned}
(aP+b)^q(cP+d) - (aP+b)(cP+d)^q &= \prod_{(\alpha,\beta)\in \mathrm{P}^1(\mathrm{F_q})} \beta(aP+b) - \alpha(cP+d) \\
&= \prod_{(\alpha,\beta)\in \mathrm{P}^1(\mathrm{F_q})} (-c\alpha + a\beta)P - (d\alpha - b\beta) \\
&= \lambda \prod_{(\alpha,\beta)\in \mathrm{P}^1(\mathrm{F_q})} \left(P - \frac{d\alpha - b\beta}{a\beta - c\alpha}\right),
\end{aligned}$$

Here $q+1$ out of the q^2+1 elements of $\{1\} \bigcup \{P + a : a \in \mathbb{F}_{q^2}\}$ occur.

Linear algebra step for P

- Each relation gives a linear equation for the logs of $\{P + a : a \in \mathbb{F}_{q^2}\}$.

- There are $\#\mathrm{PGL}(2, q^2)/\#\mathrm{PGL}(2, q) = (q^6 - q^2)/(q^3 - q) = q^3 + q$ distinct quadruples (a, b, c, d); a constant fraction of relations.

- We have a matrix of cq^3 rows (c constant) and $q^2 + 1$ columns. Heuristically, the rank is always full, so we can make a linear combination of the rows equal to $\log P$.

- Each relation brings $O(k)$ polynomials of smaller degree. The linear combination uses q^2 equations. So $\log P$ requires $O(q^2 k)$ logs.

The algorithm

We construct a descent tree in which each node is a polynomial. At each step we divide the degree by 2.

- arity of the descent tree is $O(q^2 k)$;
- height is $\log_2 k$;
- cardinality $\max(q, k)^{O(\log k)}$.

Complexity

Put $Q = q^{2k}$. When $q \approx k$ we have

$$\log Q = 2k \log q,$$

so $k = O(\log Q)$ and $q = O(\log Q)$. Then the complexity is

$$\max(q, k)^{\log k + O(1)} = (\log Q)^{O(\log \log Q)}.$$

Characteristic 2 and 3

Example

Joux computed the discrete logarithm in the field of $2^{4080} = q^{2k}$ for $q = 2^8 = k + 1$.

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When $q < k$ we embed $\mathbb{F}_{q^{2k}}$ in $\mathbb{F}_{q'^{2k}}$ with $q' = q^{\lceil \log qk \rceil}$.

Example

For $\mathbb{F}_{2^{1003}}$ we compute logs in $\mathbb{F}_{1024^{2\cdot 1003}} = \mathbb{F}_{2^{20060}}$. Complexity $\log Q^{O(\log\log Q)}$ with a larger constant.

Conclusion and open questions

- The complexity of the discrete log in $\mathbb{F}_{q^k}$ for small q was improved, replacing FFS except for a small range.
- Pairings-based cryptography in small characteristic has a much smaller complexity than expected.

Open questions:

1. The rank of the matrix in the computations is full.
2. The polynomials h_0 and h_1 can be chosen for any k and q.
3. How should one combine the various algorithms?

Thank you for your attention

Questions?